ANNIHILATING THE TATE-SHAFAREVIC GROUPS OF TATE MOTIVES

by

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A thesis presented for the degree of

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SEPTEMBER 2009

Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature.

Signature: .... Date: ....

Abstract

Let K/k be a finite abelian extension of number fields. For each odd prime p and strictly positive integer r we use the leading terms at $s = r$ of the p-adic, respectively archimedean, L-functions that are associated to K/k to define canonical ideals of $\mathbb{Q}_p[\mathrm{Gal}(K/k)]$. At each such integer r we show that Wiles' proof of the main conjecture of Iwasawa theory implies that the Tate-Shafarevic group (in the sense of Bloch and Kato) of the appropriate Tate motive is annihilated by our p-adic ideal. We also show that if the relevant special case of the Equivariant Tamagawa Number Conjecture of Burns and Flach holds, then the Tate-Shafarevic group is also annihilated by our archimedean ideal. From the first ("p-adic") result we deduce a natural analogue for totally real fields of Brumer's conjecture, and hence also of Stickelberger's theorem, an analogue at strictly positive integers of the refined Coates-Sinnott conjecture of Burns and Greither and a natural strengthening and generalisation of results of Gras and Oriat. The second ("archimedean") result extends work of Jones which itself builds upon a recent conjecture of Solomon.

In the process of obtaining these results we also prove several purely algebraic results which seem themselves to be of some independent interest. Such results include in particular a natural refinement and generalisation of a result of Snaith concerning the annihilation of the cohomology modules of a perfect complex by ideals constructed from the determinant module (in the sense of Knudsen and Mumford) of the given perfect complex.

Acknowledgements

It is a tremendous pleasure to thank my supervisor, Professor David Burns, who firstly suggested all of the problems considered in this thesis and then, with great kindness, patience and expertise, encouraged and helped me to resolve them. His meticulous proof-reading of earlier drafts of this thesis and constructive criticisms have also made the exposition immeasurably better than it would otherwise have been.

I am also grateful to David Solomon, Andrew Jones, and Andrew Parker, who's work has been built upon in many parts of this thesis. My understanding of mathematics benefited greatly from many useful conversations with them, and also with Manuel Breuning and Malcolm Bovey, both of whom went out of their way to help me on numerous occasions.

Furthermore I'm grateful to the non-academic staff of the King's College Mathematics Department, in particular Frances Benton and Rebecca Cullen without whom the department could not function.

I would like to Acknowledge the financial support of the Engineering and Physical Sciences Research Council, without which I would have been unable to undertake my PhD.

Finally I would like to thank my fiancée, Catherine Ward, for all her love, and my friends for all their support – I would not have made it through otherwise.

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Chapter 1

Introduction

The results of this thesis concern the annihilation of the Tate-Shafarevic groups (in the sense of Bloch and Kato [6]) of Tate motives by explicit ideals constructed from the leading terms of suitable L-functions.

To give a little more detail we fix a finite abelian extension of number fields K/k of group $G := \mathrm{Gal}(K/k)$ and an odd prime p and write S for the (finite) set of places of k comprising those which ramify in K/k as well as archimedean places and all places lying above p. Then the Bloch-Kato-Tate-Shafarevic group for the natural choice of p-adic integer structure on the motive $\mathbb{Q}(1)_K$ (regarded as defined over k and with a natural action of $\mathbb{Q}[G]$) is known by work of Flach [25] to be canonically isomorphic to the p-primary part of the ideal class group $\mathrm{Cl}(\mathcal{O}_K)$ of K. Furthermore for each integer r with $r > 1$ the Bloch-Kato-Tate-Shafarevic group for the natural choice of p-adic integer structure on $\mathbb{Q}(r)_K$ is equal to the Tate-Shafarevic Group in degree two of $\mathcal{O}_{K,S}$ and $\mathbb{Z}_p(r)$ that is defined by Neukirch, Schimdt and Wingberg in [45, Defn. 8.6.2] (see Lemma 3.2.4 for a proof of this fact, but the result is in principal well known). We also prove (in Lemma 3.2.7) that if K validates the Quillen-Lichtenbaum Conjecture at p, then the latter Tate-Shafarevic group is canonically isomorphic to the "p-adic Wild Kernel" of algebraic K-theory that was introduced by Banaszak in [3].

Our annihilation results regarding these Tate-Shafarevic groups are of two kinds: annihilation by elements constructed using the leading terms of p-adic L-functions (cf.

Theorems 4.1.1 and 4.1.5) and annihilation by elements constructed using the leading terms of complex L-functions (cf. Theorem 5.4.1). The techniques used in proving these two kinds of results appear to be quite different and yet are also conjecturally related (cf. Appendix B).

In a more general context the algebraic techniques developed here can also be combined with the Tamagawa number conjectures that are discussed by Fukaya and Kato in [30]. In this way one can construct explicit (conjectural) annihilators of the Bloch-Kato-Selmer modules (and hence, a fortiori, of the Bloch-Kato-Tate-Shafarevic groups) for a very wide class of motives. We note that this approach leads, amongst other things, to the formulation of a natural "strong main conjecture" for abelian varieties of the kind that Mazur and Tate explicitly ask for in [41]. This more general aspect of the theory will be described in a joint article with David Burns.

1.1 Annihilation Results Involving Leading Terms

In this section we briefly discuss the annihilation results that motivated the work in this thesis.

1.1.1 The Work of Stickelberger, of Coates and Sinnott, and of Burns and Greither

The literature is in fact replete with annihilation results for Bloch-Kato-Tate-Shafarevic groups using the values of Dirichlet L-functions at non-positive integers.

For example the classical Stickelberger Theorem states that the ideal class group of an abelian extension K of $k = \mathbb{Q}$ is annihilated by an ideal of $\mathbb{Z}[G]$ constructed from the values at $s = 0$ of partial zeta-functions.

Furthermore for any positive integer n Coates and Sinnott [18] have defined analogous ideals S_n of $\mathbb{Z}[G]$ using the values of partial zeta-functions at $s = 1 - n$. They conjectured that for all positive even n the ideal $S_n \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ annihilates the higher algebraic K-group $K_{2n-2}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ and they proved this conjecture for $n = 2$. For odd integers n Banaszak [4, Thm 1.1] has proved a weaker analogue of this conjecture. In [14] Burns and Greither formulated, and in a wide class of important cases proved, a "refined Coates-Sinnott conjecture" for an abelian CM-extension K of a totally real field k. It is known that when the Quillen-Lichtenbaum conjecture holds for K at an odd prime p, then the conjecture of Burns and Greither is indeed a natural refinement and generalisation of the Coates-Sinnott conjecture.

1.1.2 The Work of Solomon and of Jones

As far as we are aware, analogous annihilation results involving the values of partial zeta-functions at strictly positive integers were relatively unexplored until quite recently.

In [56] Solomon introduces for a finite abelian extension K of a totally real field k certain "twisted zeta-functions" whose values at $s = 0$ are related to the values at $s = 1$ of classical zeta-functions. In so doing he formulates a new Stark-type conjecture which is fundamentally p-adic in nature and involves a canonical $\mathbb{Z}_p[G]$-submodule $\mathfrak{S}_{K/k,S,p}$ of $\mathbb{Q}_p[G]$. Concerning this ideal he conjectures the following inclusion:

$$\mathfrak{S}_{K/k,S,p} \subseteq \mathbb{Z}_p[G]. \tag{1.1}$$

In his recent thesis [34] Jones proved that for any CM abelian extension K of a totally real field k, any odd prime p and any choice of field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$, the j-projection of the equivariant Tamagawa number conjecture (see §1.2.1 below) for the pair $(\mathbb{Q}(1)_K,\mathbb{Z}[G])$ implies the following natural refinement of (1.1):

$$\mathfrak{S}_{K/k,S,p} \subseteq \mathrm{Fitt}_{\mathbb{Z}[G]}\left(\mathrm{Cl}\left(\mathcal{O}_K\right)\right) \otimes_{\mathbb{Z}} \mathbb{Z}_p. \tag{1.2}$$

Jones also proved (1.2) to be valid whenever $K/\mathbb{Q}$ is abelian.

1.1.3 The Work of Gras and of Oriat

In [47] Oriat builds upon earlier work of Gras to prove an annihilation theorem of a rather different nature to those discussed above. To describe this result we assume to be given a totally real *cyclic* extension K of $\mathbb{Q}$ and an odd prime p. Then, for any faithful p-adic character ϕ of $\mathrm{Gal}(K/\mathbb{Q})$, Oriat proves that an ideal constructed from the value at $s = 1$ of the p-adic L-function of ϕ annihilates the ϕ-component of the Galois group over the cyclotomic $\mathbb{Z}_p$-extension of K of the maximal abelian p-ramified pro-p extension of K. These results of Oriat served as the original motivation for the work which led to Theorems 4.1.1 and 4.1.5.

1.2 Leading Term Conjectures

In this section we briefly recall the two types of conjectures which describe the leading terms of L-functions in terms of the determinant modules of suitable perfect complexes and will be central to this thesis. These conjectures are "Tamagawa number conjectures" which relate to the leading terms of motivic L-functions and "generalised Iwasawa main conjectures" which relate to the leading terms of p-adic L-functions.

1.2.1 The Equivariant Tamagawa Number Conjecture

The history and development of the equivariant Tamagawa number conjecture (hereinafter referred to as the "ETNC") is described by Flach in [26]. We only give a very brief summary here.

The Tamagawa number conjecture of Bloch and Kato (first set forth in 1990 [6]) is a simultaneous generalisation of both the analytic class-number formula and the celebrated conjecture of Birch and Swinnerton-Dyer. It was inspired by the computation of Tamagawa numbers of algebraic groups using motivic cohomology groups for motives of negative weight with coefficients in $\mathbb{Q}$ in place of commutative algebraic groups. The conjecture was subsequently refined by Fontaine and Perrin-Riou in [27] and [28] to apply to motives of arbitrary weight with coefficients in any commutative algebra by using the determinant construction of Knudsen and Mumford (from [38]).

The extension to the setting of motives with non-commutative coefficients was for some time hampered by the lack of a well defined determinant construction for non-commutative rings. However in 2001 Burns and Flach [11] used the virtual object formalism of Deligne [21] to formulate a Tamagawa number conjecture for an arbitrary motive M and order $\mathfrak{A}$ in the (assumed to be semisimple) coefficient algebra of M - this is the aforementioned ETNC. In this thesis the ETNC for such a pair $(M, \mathfrak{A})$ will be denoted by $\mathrm{ETNC}(M, \mathfrak{A})$. Given any prime p and field-isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ we denote the associated "j-projection" of $\mathrm{ETNC}(M, \mathfrak{A})$ by $\mathrm{ETNC}^{(j)}(M, \mathfrak{A}_p)$ (in fact when the choice of j is clear from context we will often suppress it from our notation). It is known that $\mathrm{ETNC}(M, \mathfrak{A})$ is equivalent to the validity of $\mathrm{ETNC}^{(j)}(M, \mathfrak{A}_p)$ for *every* prime p and isomorphism j.

In this thesis we fix a finite abelian extension K of a totally real base field k with $G := \mathrm{Gal}(K/k)$, an integer r, an odd prime p and an isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ and investigate $\mathrm{ETNC}^{(j)}(\mathbb{Q}(r)_K, \mathfrak{A}_p)$ for certain direct factors $\mathfrak{A}_p$ of $\mathbb{Z}_p[G]$. We recall that if $K/\mathbb{Q}$ is abelian, then $\mathrm{ETNC}^{(j)}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ is known to be valid for every p and j (by work of Burns and Greither in [15] when p is odd and $r \leq 0$, Burns and Flach in [13] when p is odd and $r \geq 1$, and by Flach in [26, Thm. 5.1] when $p = 2$) and hence that $\mathrm{ETNC}(\mathbb{Q}(r)_K, \mathbb{Z}[G])$ is itself valid in this case.

1.2.2 The Generalised Iwasawa Main Conjecture

The main conjecture of Iwasawa theory was proposed by Iwasawa in [32] and proved by Wiles in [65]. For an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, a totally real number field k in $\overline{\mathbb{Q}}$, an odd prime p and certain homomorphisms $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/k) \to \mathbb{Z}_p^\times$, Wiles's result describes the leading term at $s = 1$ of the S-truncated p-adic L-function of χ in terms of the structure of the Galois group of a particular Galois extension related to χ. This Galois group is a classical object of Iwasawa theory but can also be naturally described in terms of the continuous cohomology of the Tate module $\mathbb{Z}_p(1)$. In this way the main conjecture of Iwasawa theory can be reformulated in a manner similar to the (relevant) Tamagawa number conjecture except that the role of complex L-functions is played by p-adic L-functions. The refined (equivariant) version of the main conjecture which corresponds in a similar way to the ETNC (at least in the context of abelian extensions) was first considered by Kato in [36] where it was referred to as the "Generalised Iwasawa Main Conjecture".

1.3 Outline of the Thesis

The basic contents of this thesis is as follows.

In Chapter 2 we recall some background material and introduce some convenient notation. In Chapter 3 we recall definitions and discuss basic results concerning Galois cohomology (and various forms of Tate-Shaferevic groups) that will be used in the later parts of this thesis. In particular in §3.2.3 we recall the conjecture of Schneider which generalises Leopoldt's conjecture.

In Chapter 4 we prove a natural generalisation of a result of Oriat. In order to prove this result we deduce a generalised form of the main conjecture from Wiles' proof of the classical main conjecture and also prove a strong generalisation of an algebraic result of Snaith from [51]. We must also describe explicitly the compactly supported cohomology modules with respect to $\mathcal{O}_{K,S}$ of the Tate module $\mathbb{Z}_p(r)$ where K is a CM abelian extension of a totally real field k, r is a positive integer and p is an odd prime. Our main results in this chapter are Theorems 4.1.1 and 4.1.5 and concern the annihilation of ideal class groups, resp. Tate-Shaferevic groups, by ideals constructed using leading-terms, resp. values, of p-adic L-functions at $s = 1$, resp. $s = r$.

In Chapter 5 we prove a natural analogue of a result of Jones [34, Thm. 4.1.1] concerning the relation between $\mathrm{ETNC}(\mathbb{Q}(1)_K, \mathbb{Z}_p[G])$ and recent results of Solomon. Indeed, the main result (Theorem 5.4.1) of this chapter is that for any integer r strictly greater than one the validity of $\mathrm{ETNC}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ allows one to construct explicit annihilators of Tate-Shafarevic groups from the values of complex L-functions at $s = r$. In the course of proving this result we describe explicitly the motive $\mathbb{Q}(r)_K$ and the conjecture $\mathrm{ETNC}^{(j)}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ and construct a "higher Solomon ideal" from values of complex L-functions.

In Appendix A we prove that the "higher Solomon ideal" introduced in Chapter 5 is both independent of all of the choices made in the course of its construction and is also contained in $\mathbb{Q}_p[G]$.

In Appendix B we formulate and discuss, for each integer r strictly greater than one, a natural generalisation of "Serre's p-adic Stark conjecture at $s = 1$" which relates the leading terms of complex and p-adic L-functions at $s = r$.

Chapter 2

Preliminaries

2.1 Algebraic Preliminaries

2.1.1 Notation

We fix an odd prime p and write $\mathbb{Z}_p$ and $\mathbb{Q}_p$ for the ring of p-adic integers and the field of p-adic rationals respectively. We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and write $\mathbb{C}_p$ for the completion of $\overline{\mathbb{Q}}_p$. We set $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

For each positive integer n let μ_{p^n} denote the group of $p^{n\text{th}}$-roots of unity in $\overline{\mathbb{Q}}$. For positive integers n and s we define left $G_{\mathbb{Q}}$-modules by setting $\mathbb{Z}/p^n\mathbb{Z}(s) := \mu_{p^n}^{\otimes s}$ and $\mathbb{Z}/p^n\mathbb{Z}(-s) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^n\mathbb{Z}(s), \mathbb{Z}/p^n\mathbb{Z})$, the latter endowed with the natural contragredient (left) action of $G_{\mathbb{Q}}$ given by $\sigma \cdot \theta(\zeta) = \theta(\sigma^{-1}(\zeta))$ for $\sigma \in G_{\mathbb{Q}}$ and $\zeta \in \mu_{p^n}^{\otimes s}$. We also define $\mathbb{Z}_p(s) := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}(s)$ and $\mathbb{Q}_p/\mathbb{Z}_p(s) := \lim_{\rightarrow n} \mathbb{Z}/p^n\mathbb{Z}(s)$. For any $\mathbb{Z}_p$-module M that has an action of $G_{\mathbb{Q}}$ we let $M^\vee$ denote the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ and M^* the linear dual $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$, each endowed with the contragredient $G_{\mathbb{Q}}$-action. In particular we note that for each integer s there is a natural isomorphism of $G_{\mathbb{Q}}$-modules $\mathbb{Z}_p(s) \cong (\mathbb{Q}_p/\mathbb{Z}_p(-s))^\vee$. If the $G_{\mathbb{Q}}$-module M also has a commuting action of $\mathbb{Z}_p[G]$ for some finite abelian group G, then it is convenient for us to endow the dual $G_{\mathbb{Q}}$-modules $M^\vee$ and M^* with the commuting action of $\mathbb{Z}_p[G]$ given by $g \cdot \theta(m) = \theta(gm)$ for all $g \in G$ and $m \in M$ (note that this is not the usual contragredient action of G but is nevertheless a valid left action since

G is abelian).

Given any $\mathbb{Z}_p$-module we write $M_{\rm tor}$ for the $\mathbb{Z}_p$-torsion submodule of M, set $M_{\rm tf} := M/M_{\rm tor}$ and define the $\mathbb{Z}_p$-cotorsion part $M_{\rm cotor}$ of M to be the quotient of M by its maximal divisible subgroup. We often use the fact that the module $(M^\vee)_{\rm cotor}$ is canonically isomorphic to $(M_{\rm tor})^\vee$.

For any group G and G-module N the submodule N^G of N is the maximal submodule of N upon which the action of G is trivial. We also write N_G for the G-coinvariants of N, the maximal quotient of N upon which G acts trivially.

If G is a finite abelian group, then we shall write χ for both a complex, resp. $\mathbb{C}_p$, character of G and also for the induced ring homomorphism $\mathbb{C}[G] \to \mathbb{C}$, resp. $\mathbb{C}_p[G] \to \mathbb{C}_p$. We also use (often without explicit comment) the fact that taken together these homomorphisms induce canonical ring isomorphisms $\mathbb{C}[G] \cong \prod_\chi \mathbb{C}$ and $\mathbb{C}_p[G] \cong \prod_\phi \mathbb{C}_p$ where χ runs over all irreducible complex characters of G and ϕ over all irreducible $\mathbb{C}_p$ characters of G. Given any such character χ of G we write e_χ for the associated idempotent $\frac{1}{|G|}\sum_{g\in G}\chi(g)g^{-1}$ of $\mathbb{C}[G]$, resp. $\mathbb{C}_p[G]$. We also write $\chi_{0,G}$ for the trivial character of G and set $e_G := e_{\chi_{0,G}}$.

For any non-negative integer n and associative unital ring $\mathfrak{A}$ we write $K_n(\mathfrak{A})$ for the algebraic K-group in degree n of Quillen as described by Srinivas in [58, §2]. In particular we recall that for $0 \leq n \leq 2$ this group agrees with the low-degree algebraic K-groups defined, for example, in [44], [60], or [37].

2.1.2 Perfect Complexes and Determinant Modules

The determinant construction of Grothendieck, Knudsen and Mumford (cf. [38, §1]) will be used throughout this thesis. The reader is referred to *loc. cit.* for full details of this construction but for convenience we also give a brief summary below.

Let $\mathfrak{A}$ be any commutative unital Noetherian ring that decomposes as a product of local rings $\mathfrak{A} = \prod_{i \in I} \mathfrak{A}_i$ for some finite indexing set I. This decomposition induces a corresponding decomposition for any finitely generated $\mathfrak{A}$-module $M := \bigoplus_{i \in I} M_i$ and M is then projective if and only if M_i is a free $\mathfrak{A}_i$-module for all i in I (cf. [23, Thm. A3.3]). Also recall that for any positive integer n the n^{th} exterior power $\bigwedge_{\mathfrak{A}}^n M$ of an $\mathfrak{A}$-module M is such that if M is free of rank n then each choice of basis of M induces an isomorphism of $\mathfrak{A}$-modules $\bigwedge_{\mathfrak{A}}^n M \cong \mathfrak{A}$ (cf. [39, Chapter XIX]).

Definition 2.1.1. Given any finitely generated projective $\mathfrak{A}$-module $P := \bigoplus_{i \in I} P_i$ for each index $i \in I$ we write $\mathrm{rk}_{\mathfrak{A}_i}(P_i)$ for the rank of the (free) $\mathfrak{A}_i$-module P_i. We write $\mathrm{rk}_{\mathfrak{A}}(P)$ for the function $I \to \mathbb{Z}$ which maps each i in I to $\mathrm{rk}_{\mathfrak{A}_i}(P_i)$.

Definition 2.1.2. A *graded invertible* $\mathfrak{A}$*-module* is a pair (M, f) comprising a free rank one $\mathfrak{A}$-module M and a function $f : I \to \mathbb{Z}$. From any two such pairs (M, f) and (N, g) we obtain additional graded invertible $\mathfrak{A}$-modules by setting $(M, f) \otimes_{\mathfrak{A}} (N, g) := (M \otimes_{\mathfrak{A}} N, f + g)$ and $(M, f)^{-1} := (\mathrm{Hom}_{\mathfrak{A}}(M, \mathfrak{A}), -f)$.

For a finitely generated projective $\mathfrak{A}$-module $P := \bigoplus_{i \in I} P_i$ we define the *determinant module* of P to be the graded invertible $\mathfrak{A}$-module

$$[P]_{\mathfrak{A}} := \left(\bigoplus_{i \in I} \left(\bigwedge\nolimits_{\mathfrak{A}_i}^{\mathrm{rk}_{\mathfrak{A}_i}(P_i)} P_i \right), \mathrm{rk}_{\mathfrak{A}}(P) \right).$$

For a bounded complex of finitely generated projective $\mathfrak{A}$-modules

$$P^\bullet : 0 \to P^a \to \cdots \to P^b \to 0,$$

in which the term P^a occurs in degree a, we define the $\mathfrak{A}$-*determinant module* $[P^\bullet]_{\mathfrak{A}}$

of $P^\bullet$ by setting

$$[P^\bullet]_{\mathfrak{A}} := \bigotimes_{n\in\mathbb{Z}} [P^n]_{\mathfrak{A}}^{(-1)^n}.$$

We recall that the module $[P^\bullet]_{\mathfrak{A}}$ depends (up to unique isomorphism) only upon the quasi-isomorphism class of $P^\bullet$ (cf. [38, Thm. 1]).

Definition 2.1.3. We call a complex $C^\bullet$ of $\mathfrak{A}$-modules *perfect* if it is quasi-isomorphic to a *bounded* complex of finitely generated projective $\mathfrak{A}$-modules. We call an $\mathfrak{A}$-module M *perfect* if it is both finitely generated and of finite projective dimension (ie. if the associated complex $M[0]$ is perfect). We denote by $\mathrm{PMod}(\mathfrak{A})$ the category of perfect complexes of $\mathfrak{A}$-modules and write $\mathcal{D}^p(\mathfrak{A})$ for the derived category of $\mathrm{PMod}(\mathfrak{A})$.

We call a complex of $\mathfrak{A}$-modules *cohomologically-perfect* if it is acyclic outside a finite number of degrees and all of its cohomology groups are perfect $\mathfrak{A}$-modules. It is well known (and straightforward to show) that a cohomologically-perfect complex is automatically a perfect complex.

Given a perfect complex $C^\bullet$ we may choose a bounded complex $P^\bullet$ of finitely generated projectives that is quasi-isomorphic to $C^\bullet$ and set $[C^\bullet]_{\mathfrak{A}} := [P^\bullet]_{\mathfrak{A}}$. Then the final sentence of Definition 2.1.2 implies that, up to unique isomorphism, $[C^\bullet]_{\mathfrak{A}}$ depends only upon $C^\bullet$. In particular given any perfect $\mathfrak{A}$-module M we may set $[M]_{\mathfrak{A}} := [M[0]]_{\mathfrak{A}}$.

We shall often make use of the following result.

Lemma 2.1.4. ([38, Thm. 2, Rem. b]) *Given any cohomologically-perfect complex $C^\bullet$ of $\mathfrak{A}$-modules there is a canonical isomorphism of graded invertible $\mathfrak{A}$-modules*

$$[C^\bullet]_{\mathfrak{A}} \cong \bigotimes_{n\in\mathbb{Z}} [H^n(C^\bullet)]_{\mathfrak{A}}^{(-1)^n}.$$

Finally we recall that for any commutative unital ring $\mathfrak{A}$ and any perfect complex $C^\bullet$ of $\mathfrak{A}$-modules we may define the *(projective) Euler characteristic* of $C^\bullet$ in the following way. We fix a bounded complex of finitely generated projective $\mathfrak{A}$-modules $P^\bullet$ that is quasi-isomorphic to $C^\bullet$ and define the Euler characteristic of $C^\bullet$ to be the element $\sum_{n\in\mathbb{Z}}(-1)^n[P^n]$ of $K_0(\mathfrak{A})$. This element is easily shown to be independent of the choice of $P^\bullet$.

2.1.3 Fitting Ideals

Let $\mathfrak{A}$ be a commutative unital Noetherian ring.

Definition 2.1.5. Given a finitely generated $\mathfrak{A}$-module M we choose an exact sequence of $\mathfrak{A}$-modules of the form

$$\mathfrak{A}^b \xrightarrow{f} \mathfrak{A}^a \to M \to 0. \tag{2.1}$$

After fixing an $\mathfrak{A}$-basis of $\mathfrak{A}^a$ we identify $\wedge^a_{\mathfrak{A}}(\mathfrak{A}^a)$ with $\mathfrak{A}$ and then define the $\mathfrak{A}$-Fitting ideal (also known as the *initial* $\mathfrak{A}$*-Fitting invariant*) of M to be the ideal

$$\mathrm{Fitt}_{\mathfrak{A}}(M) := \mathrm{Im}(\wedge^a_{\mathfrak{A}} f) \subseteq \mathfrak{A}.$$

Then $\mathrm{Fitt}_{\mathfrak{A}}(M)$ coincides with the ideal of $\mathfrak{A}$ generated by all $a\times a$-minors of the matrix of f with respect to some chosen $\mathfrak{A}$-bases of $\mathfrak{A}^b$ and $\mathfrak{A}^a$ (and hence is independent of the choice of these bases). It is also possible to show that $\mathrm{Fitt}_{\mathfrak{A}}(M)$ is independent of the choice of free resolution (2.1) (cf. [46, Thm. 1]).

Lemma 2.1.6. *Let M be a finitely generated $\mathfrak{A}$-module.*

(i) Then $\mathrm{Fitt}_{\mathfrak{A}}(M) \subseteq \mathrm{Ann}_{\mathfrak{A}}(M)$.

(ii) *Let $\mathfrak{A}$ be a finitely generated $\mathbb{Z}_p$-algebra (which therefore satisfies the conditions of §2.1.2) and set $A := \mathfrak{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If M is both finite and has projective dimension at most one, then $\mathrm{Fitt}_{\mathfrak{A}}(M)$ is an invertible $\mathfrak{A}$-module and the canonical isomorphism of graded invertible A-modules $[M[0] \otimes_{\mathfrak{A}} A]_A^{-1} \cong [0]_A^{-1} = (A, 0)$ induces an equality $[M[0]]_{\mathfrak{A}}^{-1} = (\mathrm{Fitt}_{\mathfrak{A}}(M), 0)$.*

Proof. Claim (i) is well known.

Regarding claim (ii) we note that if M is a finitely generated $\mathfrak{A}$-module of projective dimension at most one, then there is an integer a and a surjection $\mathfrak{A}^a \twoheadrightarrow M$ with projective kernel P. If M is also finite, then, since $\mathfrak{A}$ is as in §2.1.2, P is itself a free $\mathfrak{A}$-module of rank a. We therefore obtain a complex of $\mathfrak{A}$-modules of the form $\mathfrak{A}^a \xrightarrow{f} \mathfrak{A}^a$ (concentrated in degrees -1 and 0) that is quasi-isomorphic to $M[0]$. This implies that the ideal $\mathrm{Fitt}_{\mathfrak{A}}(M)$ is generated by the element $\det(f) \in A^\times$ and so is an invertible $\mathfrak{A}$-module. Further, by definition of the determinant module, one has $[M[0]]_{\mathfrak{A}} = [\mathfrak{A}^a]_{\mathfrak{A}} \otimes_{\mathfrak{A}} [\mathfrak{A}^a]_{\mathfrak{A}}^{-1}$. There are therefore canonical isomorphisms

$$
\begin{aligned}
[M[0]]_{\mathfrak{A}}^{-1} &= [\mathfrak{A}^a]_{\mathfrak{A}}^{-1} \otimes_{\mathfrak{A}} [\mathfrak{A}^a]_{\mathfrak{A}} \\
&\cong (\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A}), -a) \otimes_{\mathfrak{A}} [\mathrm{Im}(f)]_{\mathfrak{A}} \\
&\cong (\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A}), -a) \otimes_{\mathfrak{A}} (\textstyle\bigwedge_{\mathfrak{A}}^{a} \mathrm{Im}(f), a) \\
&= (\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A}), -a) \otimes_{\mathfrak{A}} (\mathrm{Im}(\wedge_{\mathfrak{A}}^{a} f), a) \\
&= (\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A}), -a) \otimes_{\mathfrak{A}} (\mathfrak{A}, a) \otimes_{\mathfrak{A}} (\mathrm{Im}(\wedge_{\mathfrak{A}}^{a} f), 0) \\
&\cong (\mathrm{Fitt}_{\mathfrak{A}}(M), 0).
\end{aligned}
\tag{2.2}
$$

Here the first isomorphism uses the definition of $[\mathfrak{A}^a]_{\mathfrak{A}}^{-1}$, the natural isomorphism $\bigwedge_{\mathfrak{A}}^{a} \mathfrak{A}^a \cong \mathfrak{A}$ and the isomorphism $\mathfrak{A}^a \cong \mathrm{Im}(f)$ induced by f; the second isomorphism comes from the fact that $\mathrm{Im}(f)$ has rank a; the third equality is by definition of the tensor product of graded invertible $\mathfrak{A}$-modules; the final isomorphism is induced by the isomorphism $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A},\mathfrak{A}) \otimes_{\mathfrak{A}} \mathfrak{A} \cong \mathfrak{A}$ and the definition of $\mathrm{Fitt}_{\mathfrak{A}}(M)$. It therefore suffices to note that (as can be checked explicitly) the scalar extension of (2.2) agrees with the isomorphism $[M[0] \otimes_{\mathfrak{A}} A]_A^{-1} \cong [0]_A^{-1} = (A, 0)$ that is induced by the acyclicity of the complex $M[0] \otimes_{\mathfrak{A}} A$. $\square$

Lemma 2.1.7. *Let $\mathfrak{A} = \mathbb{Z}_p[G]$ and $A := \mathbb{Q}_p[G]$ for a finite abelian group G and fix a finitely generated $\mathfrak{A}$-module M.*

(i) For any idempotent e of A one has $\mathrm{Fitt}_{\mathfrak{A}e}(M \otimes_{\mathfrak{A}} \mathfrak{A}e) = \mathrm{Fitt}_{\mathfrak{A}}(M)e$.

(ii) Define e_M to be the sum of all primitive idempotents of A that annihilate $M \otimes_{\mathfrak{A}} A$. Then $\mathrm{Fitt}_{\mathfrak{A}e_M}(M \otimes_{\mathfrak{A}} \mathfrak{A}e_M) = \mathrm{Fitt}_{\mathfrak{A}}(M)e_M = \mathrm{Fitt}_{\mathfrak{A}}(M) \subseteq \mathfrak{A} \cap \mathfrak{A}e_M$.

Proof. After fixing a free resolution of M as in (2.1) we obtain a free resolution of $\mathfrak{A}e$-modules $(\mathfrak{A}e)^b \xrightarrow{\hat{f}} (\mathfrak{A}e)^a \to M \otimes_{\mathfrak{A}} \mathfrak{A}e \to 0$ (since the functor $- \otimes_{\mathfrak{A}} \mathfrak{A}e$ is right-exact). The image of $\hat{f}$ is equal to the projection of $\mathrm{Im}(f)$ from $\mathfrak{A}$ to $\mathfrak{A}e$. Hence by the definition of the Fitting ideal one sees that $\mathrm{Fitt}_{\mathfrak{A}e}(M \otimes_{\mathfrak{A}} \mathfrak{A}e)$ is equal to the projection of $\mathrm{Fitt}_{\mathfrak{A}}(M)$ in $\mathfrak{A}e$. Hence claim (i) follows.

To prove claim (ii) it suffices to prove that $\mathrm{Fitt}_{\mathfrak{A}}(M) = \mathrm{Fitt}_{\mathfrak{A}}(M)e_M$. For each x in $\mathrm{Fitt}_{\mathfrak{A}}(M) \subseteq \mathfrak{A}$ one has $x = xe_M + x(1-e_M) \in A$. Given any $\mathbb{Q}_p$-character χ of G we set $e_\chi := \frac{1}{|G|}\sum_{g\in G}\chi(g)g^{-1} \in A$. Now $1-e_M = \sum e_\chi$ where χ runs over all irreducible $\mathbb{Q}_p$-characters of G for which $(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)e_\chi$ is non-zero and so it suffices to prove that $\mathrm{Fitt}_{\mathfrak{A}}(M)e_\chi = 0$ for all such χ. But for any such χ the sequence (2.1) induces a free resolution of $\mathfrak{A}e_\chi$-modules of the form $(\mathfrak{A}e_\chi)^b \xrightarrow{\hat{f}} (\mathfrak{A}e_\chi)^a \to M \otimes_{\mathfrak{A}} \mathfrak{A}e_\chi \to 0$ and since, by assumption, e_χ does not annihilate $M \otimes_{\mathfrak{A}} A$ the $\mathfrak{A}e_\chi$-module $M \otimes_{\mathfrak{A}} \mathfrak{A}e_\chi$ has rank at least one. It follows that the $\mathfrak{A}e_\chi$-module $\mathrm{Im}(\hat{f})$ has rank at most $a-1$ and hence that $\mathrm{Fitt}_{\mathfrak{A}e_\chi}(M \otimes_{\mathfrak{A}} \mathfrak{A}e_\chi) = \mathrm{Im}(\wedge^a_{\mathfrak{A}e_\chi} \hat{f})$ is a torsion submodule of $\mathfrak{A}e_\chi$. Since $\mathfrak{A}e_\chi$ is torsion-free one therefore has $\mathrm{Fitt}_{\mathfrak{A}e_\chi}(M \otimes_{\mathfrak{A}} \mathfrak{A}e_\chi) = 0$ and hence, by claim (i), also $\mathrm{Fitt}_{\mathfrak{A}}(M)e_\chi = 0$, as required. □

Lemma 2.1.8. *(cf. [52, Prop. 2.8]) Let G be a finite abelian group and for each $\mathbb{Z}_p[G]$-module M endow $M^\vee$ with the G-action described in §2.1.1. Then one has $\mathrm{Fitt}_{\mathbb{Z}_p[G]}(M) = \mathrm{Fitt}_{\mathbb{Z}_p[G]}(M^\vee)$ for every finitely generated $\mathbb{Z}_p[G]$-module M if and only if the Sylow-p-subgroup of G is cyclic.*

2.2 Field Theoretic Preliminaries

2.2.1 Notation

For any field K and any place w of K we denote by K_w the completion of K at w. We write $\overline{K}$ for a fixed separable closure of K.

For any finite Galois extension K/k of number fields with $K \subset \overline{k}$ and any set of places Σ of k we denote by $\Sigma(K)$ the set of places of K lying above those in Σ. If Σ contains all archimedean places of k, then we write $\mathcal{O}_{K,\Sigma}$ for the subring of K comprising all elements that are integral at all places outside $\Sigma(K)$. We also use the following notation.

- $G_{K/k} := \mathrm{Gal}(K/k)$.
- $G_K := \mathrm{Gal}(\overline{K}/K)$ and $G_k := \mathrm{Gal}(\overline{K}/k) = \mathrm{Gal}(\overline{k}/k)$.
- $S^0_{K/k}$ is the finite set of places of k comprising those which ramify in K/k and all archimedean places.
- $S_{K/k}$ is the union of $S^0_{K/k}$ and the set $\{v \mid p\}$ of places of k which lie above p.

- K^∞ is the cyclotomic $\mathbb{Z}_p$-extension of K.

- Given any finite set of places Σ of k containing all places above p let $M_\Sigma(K)$ be the maximal abelian extension of K that is unramified outside of $\Sigma(K)$ and $M_\Sigma^p(K)$ the maximal abelian pro-p extension of K unramified outside of $\Sigma(K)$.

- Given any finite set of places Σ of k containing all places above p set $G_{K,\Sigma} := \mathrm{Gal}(M_\Sigma(K)/K)$, $\mathcal{G}_{K,\Sigma} := \mathrm{Gal}(M_\Sigma^p(K)/K)$ and $\mathcal{H}_{K,\Sigma} := \mathrm{Gal}(M_\Sigma^p(K)/K^\infty)$.

When the field extension is clear from context we will often suppress the subscripts. In particular we shall sometimes write S for $S_{K/k}$, $\mathcal{G}$ for $\mathcal{G}_{K,S_{K/k}}$ and $\mathcal{H}$ for $\mathcal{H}_{K,S_{K/k}}$.

For any place w of K we let $G_w \subseteq G_K$ be the decomposition group associated to any choice of place of $\overline{K}$ lying above w. If w is non-archimedean we also write

- I_w for the inertia subgroup of w in G_w, and

- f_w for the Frobenius automorphism in G_w/I_w.

The group $G_{K,S}$ (and hence also G_K and each G_w) acts naturally on the group of p-power-roots of unity $\mu_{p^\infty}(\overline{K})$ in $\overline{K}$ and we define the cyclotomic character $\chi_{\mathrm{cyclo}} : G_{K,S} \to \mathbb{Z}_p^\times$ to be such that if ζ belongs to $\mu_{p^\infty}(\overline{K})$ then $g(\zeta) = \zeta^{\chi_{\mathrm{cyclo}}(g)}$ for every g in $G_{K,S}$. We note that for each integer r the $G_{K,S}$-modules $\mathbb{Z}_p(r)$, $\mathbb{Q}_p(r)$ and $\mathbb{Q}_p/\mathbb{Z}_p(r)$ identify with the groups $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$ respectively, upon which each g in $G_{K,S}$ acts as multiplication by $\chi_{\mathrm{cyclo}}(g)^r$ (cf. [45, Defn. 7.3.6]).

If K is any finite abelian extension of k in $\overline{k}$ that is unramified outside S, then for each integer r we write $\mathbb{Z}_p(r)_K$ for the $G_{k,S} \times G$-module $\mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G]$ upon which G acts by multiplication on $\mathbb{Z}_p[G]$ and $G_{k,S}$ acts diagonally (on the second term via the natural projection $G_k \to G$). We also then set $\mathbb{Q}_p(r)_K := \mathbb{Z}_p(r)_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p(r)_K := \mathbb{Z}_p(r)_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Q}_p(r)_K/\mathbb{Z}_p(r)_K$.

For any set Σ of places of K we denote by Y_Σ the free abelian group on the places in Σ. For any positive integer s we let $Y_{K,s}$ denote the $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module

$$Y_{K,s} := \bigoplus_{K \hookrightarrow \mathbb{C}} (2\pi i)^s \mathbb{Z},$$

upon which G acts via pre-composition with the indexing maps and $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts diagonally via post-composition with the indexing maps and the natural action on the coefficients. We write $Y_{K,s}^+$ for the associated G-module of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants $(Y_{K,s})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$.

For any number field K and prime p we write $\mu(K,p)$ for Iwasawa's "μ-invariant" for K and p as defined in [33]. We recall that Iwasawa has conjectured that $\mu(K,p)$ is always equal to 0 and that this has been proved by Ferrero and Washington [24] in the case that $K/\mathbb{Q}$ is abelian.

2.2.2 CM Extensions

Let K be a CM abelian extension of a totally real field k. For convenience we introduce the following notation:

- K^+ is the maximal totally real sub-field of K.
- $G_{K/k}^+ := G_{K^+/k}$.
- $\tau = \tau_K$ is the unique non-trivial element of the subgroup $\mathrm{Gal}(K/K^+) \subsetneq G_{K/k}$.

As before when there is no chance of confusion certain subscripts will be suppressed. In particular we shall write S for $S_{K/k}$.

The focus of this thesis is on generalising results that were originally formulated to relate to the values of L-functions at $s = 1$ rather than $s = 0$. For this reason it is convenient to modify the usual notation for the parity idempotents $e^+ = \frac{1+\tau}{2}$ and $e^- = \frac{1-\tau}{2}$ in $\mathbb{Z}[\frac{1}{2}][G]$.

Definition 2.2.1. For any integer r we define

$$e_r^{\pm} := \frac{1 \mp (-1)^r \tau}{2} \in \mathbb{Z}[\tfrac{1}{2}][G] \subseteq \mathbb{Z}_p[G],$$

(where the last inclusion follows from the fact that p is odd).

We note that for any $\mathbb{Z}_p[G]$-module M, the submodule $Me_r^{\pm}$ is naturally a $\mathbb{Z}_p[G]e_r^{\pm}$-module and one has $M = Me_r^+ \oplus Me_r^-$. In the sequel statements referring to $\mathbb{Z}_p[G]e_r^{\pm}$ and $\mathbb{Z}_p[G]e_r^{\pm}$-modules will often be referred to as statements about the "plus/minus-part" as appropriate.

Remark 2.2.2. When r is odd, resp. even, one has $e_r^{\pm} = e^{\pm}$, resp. $e_r^{\pm} = e^{\mp}$. For any integer r it is also clear that $e_r^{\pm} = e_{1-r}^{\mp}$.

Remark 2.2.3. Remark 2.2.2 implies that the restriction map $G \to G^+$ induces a canonical isomorphism of rings

$$\mathbb{Z}_p\big[G^+\big] \cong \begin{cases} \mathbb{Z}_p[G]e_r^+ & \text{if } r \text{ is odd} \\ \mathbb{Z}_p[G]e_r^- & \text{if } r \text{ is even.} \end{cases}$$

Remark 2.2.4. Given any finite abelian ramified extension L/k of totally real fields we may choose a quadratic imaginary extension E of k that ramifies only at the archimedean places and at (some of) the places that ramify in L/k. Then the field $K := EL$ is a CM abelian extension of k for which $K^+ = L$ and $S_{K/k} = S_{L/k}$. Hence via the isomorphism of Remark 2.2.3 if r is odd (resp. even) there is a one-to-one correspondence between statements regarding Galois module structure for finite abelian extensions of totally real fields and statements regarding Galois module structure for the "plus-part" (resp. "minus-part") of CM abelian extensions over totally real fields.

Lemma 2.2.5. *Given any integer r one has a canonical isomorphism of $\mathbb{Z}_p[G^+]$-modules*

$$\mathcal{G}_{K^+,S} \cong \begin{cases} e_r^+\mathcal{G}_{K,S} & \text{if } r \text{ is odd} \\ e_r^-\mathcal{G}_{K,S} & \text{if } r \text{ is even.} \end{cases}$$

Proof. It suffices to prove that $e^+\mathcal{G}_{K,S}$ is naturally isomorphic to $\mathcal{G}_{K^+,S}$. Now since τ acts on $\mathcal{G}_{K,S}$ by conjugation the quotient $e^+\mathcal{G}_{K,S} \cong \mathcal{G}_{K,S}/(1-\tau)\mathcal{G}_{K,S}$ is equal to $\mathrm{Gal}(L/K)$ where L is the maximal extension of K inside $M_S^p(K)$ that is abelian over K^+. It is therefore clear that $M_S^p(K^+)K \subseteq L$, that L is CM and that the maximal real subfield L^+ of L is a pro-p abelian extension of K^+ that is unramified outside S and hence that $L^+ \subseteq M_S^p(K^+)$. Putting these facts together one finds that $L^+ = M_S^p(K^+)$ and $L = KM_S^p(K^+)$ and hence that the kernel $\mathrm{Gal}(L/M_S^p(K^+))$ of the natural restriction map $\mathrm{Gal}(L/K^+) \to \mathcal{G}_{K^+,S}$ has order 2. Thus, since $\mathrm{Gal}(L/K)$ has index 2 in $\mathrm{Gal}(L/K^+)$, the composite homomorphism $e^+\mathcal{G}_{K,S} \cong \mathrm{Gal}(L/K) \subset \mathrm{Gal}(L/K^+) \to \mathcal{G}_{K^+,S}$ is bijective. □

2.2.3 L-functions

In this subsection we fix a finite abelian extension of number fields K/k and set $G := \mathrm{Gal}(K/k)$. We also fix a finite set of places Σ of k that contains all archimedean places.

We regard the canonical ring isomorphisms $\mathbb{C}[G] \cong \prod_\chi \mathbb{C}$ and $\mathbb{C}_p[G] \cong \prod_\phi \mathbb{C}_p$ that are discussed in §2.1.1 as identifications.

Definition 2.2.6. For each finite dimensional complex character χ of G with representation space V_χ and for each place w of K above a place v of k the Frobenius element f_w acts on the subspace $V_\chi^{I_w}$. On the half plane $\{s \in \mathbb{C} \mid \Re(s) > 1\}$ we can therefore define the Σ-truncated Artin L-function of χ by setting

$$L_\Sigma(\chi, s) := \prod_{v \notin \Sigma} \det_{\mathbb{C}}(1 - f_w \mathrm{N}v^{-s} \mid V_\chi^{I_w})^{-1}$$

(cf. [63]). We recall that there exists a unique meromorphic continuation of $L_\Sigma(\chi, s)$ to the entire complex plane.

We also define the Σ-truncated equivariant Artin L-function of K/k to be the unique meromorphic $\mathbb{C}[G]$-valued function of the complex variable s obtained by setting

$$L_{K/k,\Sigma}(s) = \left(L_\Sigma(\chi, s)\right)_\chi .$$

Definition 2.2.7. For each finite dimensional $\mathbb{C}_p$ character ϕ of G the Σ-truncated p-adic L-function of ϕ is the unique p-adic meromorphic function $L_{p,\Sigma}(\phi, \cdot) : \mathbb{Z}_p \to \mathbb{C}_p$ such that for each strictly negative integer n and each field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ one has

$$L_{p,\Sigma}(\phi, n) = j\left(L_\Sigma(j^{-1}(\phi \cdot \omega^{n-1}), n)\right)$$

where $\omega : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ is the Teichmüller character (cf. [16, §5.2] or [62, Chap. VI, §2]).

We also define the Σ-truncated equivariant p-adic L-function to be the associated meromorphic $\mathbb{C}_p[G]$-valued function

$$L_{p,\Sigma}(s) = (L_{p,\Sigma}(\phi, s))_\phi .$$

It will also be convenient to introduce the following slightly modified version of the latter function.

Definition 2.2.8. We define the Σ-truncated "twisted" equivariant p-adic L-function to be the meromorphic $\mathbb{C}_p[G]$-valued function

$$\mathfrak{L}_{p,\Sigma}(s) = \left(L_{p,\Sigma}(\omega^{1-s} \cdot \phi, s)\right)_\phi .$$

Remark 2.2.9. An explicit comparison of definitions makes it clear that for each strictly negative integer n, each field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ and each $\mathbb{C}_p$ character ϕ of G one has

$$\phi(\mathfrak{L}_{p,\Sigma}(n)) = L_{p,\Sigma}(\omega^{1-n} \cdot \phi, n) = j(L_\Sigma(j^{-1} \circ \phi, n)).$$

In Appendix B we shall in fact conjecture a precise relationship between the values of these archimedean and p-adic L-functions at strictly positive integers.

For any meromorphic function f of either a complex or p-adic variable s and any complex number r we will use $f^*(r)$ to denote the leading term of the Taylor expansion of f around $s = r$.

Chapter 3

Galois Cohomology and Tate-Shafarevic Groups

3.1 Galois Cohomology

We first introduce some convenient notation which will be used throughout this thesis.

Definition 3.1.1.

- Given a cochain complex denoted by $R\Gamma_*(\mathcal{R}, N)$ for some symbol $*$, ring $\mathcal{R}$ and module N we write $H^n_*(\mathcal{R}, N)$ for the cohomology group of the given complex in degree n.

- Given a symbol $*$, an integer r and a cochain complex $R\Gamma_*(\mathcal{R}, \mathbb{Z}_p(r))$ of $\mathbb{Z}_p$-modules we define

$$R\Gamma_*(\mathcal{R}, \mathbb{Q}_p(r)) := R\Gamma_*(\mathcal{R}, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(r),$$

and

$$R\Gamma_*(\mathcal{R}, \mathbb{C}_p(r)) := R\Gamma_*(\mathcal{R}, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p(r).$$

We note that if $R\Gamma_*(\mathcal{R}, \mathbb{Z}_p(r))$ has an action of some group G then there is an induced G-action on both $R\Gamma_*(\mathcal{R}, \mathbb{Q}_p(r))$ and $R\Gamma_*(\mathcal{R}, \mathbb{C}_p(r))$.

- Let K/k be a finite abelian extension of number fields and Σ a finite set of places of k. Let N be a topological G_w-module for each place w of K lying above a place in Σ. Given an integer n, a symbol $*$, and for each $w \in \Sigma(K)$ a group $H^n_*(K_w, N)$ we define

$$P^n_*(\mathcal{O}_{K,\Sigma}, N) := \bigoplus_{w \in \Sigma(K)} H^n_*(K_w, N).$$

3.1.1 Continuous Cochain Cohomology

For a commutative unital ring $\mathfrak{A}$ the basic definitions and properties of cochain complexes of $\mathfrak{A}$-modules can be found, for example, in [64, §1.1]. For a pro-finite group G and topological G-module N the basic definitions and properties of the continuous homogeneous cochain complex $C^\bullet(G, N)$ that we will use can be found in [45, Chap. II, §1-2].

Henceforth let K/k be a finite abelian extension of number fields with $G := \mathrm{Gal}(K/k)$ and write S for the (finite) set consisting of all places of k which ramify in K/k, all archimedean places of k, and all places of k lying above p. Recall that $G_{K,S}$ denotes the galois group over K of the maximal extension of K inside $\overline{K}$ that is unramified outside of $S(K)$. Recall also that for each place w of K the decomposition group corresponding to some place of G_K lying above w is denoted by G_w.

Definition 3.1.2. For any topological $G_{K,S}$-module N we set

$$R\Gamma(\mathcal{O}_{K,S}, N) := C^\bullet(G_{K,S}, N).$$

Given any place w of K and any G_w-module N we also set

$$R\Gamma(K_w, N) := C^\bullet(G_w, N).$$

Remark 3.1.3. A topological $G_{K,S}$-module N is a topological G_K-module (via the forgetful functor) and hence also a topological G_w-module (by restriction). Thus for any such module N and any integer n the group $P^n(\mathcal{O}_{K,S}, N)$ introduced in Definition 3.1.1 is well-defined.

Remark 3.1.4. For all integers n and r the natural conjugation action of G on $G_{K,S}$ induces an action of $\mathbb{Z}_p[G]$ on each of the groups $H^n(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$, $H^n(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r))$, $P^n(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ and $P^n(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r))$.

3.1.2 Compact Support Cohomology

Definition 3.1.5. Given any topological $G_{K,S}$-module N we follow Burns and Flach [11] in defining the *compact support cohomology complex* to be a complex of $G_{K,S}$-modules which lies in an exact triangle

$$R\Gamma_c(\mathcal{O}_{K,S}, N) \to R\Gamma(\mathcal{O}_{K,S}, N) \to \bigoplus_{w \in S(K)} R\Gamma(K_w, N) \to \tag{3.1}$$

in which the second arrow is induced by the natural localisation maps.

In the sequel we shall make much use of the following easy observation.

Lemma 3.1.6. *Given any topological $G_{K,S}$-module N the compact support cohomology group $H^0_c(\mathcal{O}_{K,S}, N)$ is zero.*

Proof. For each place w in $S(K)$ it is an immediate consequence of the definition of the complex $R\Gamma(K_w, N)$ that the module $H^{-1}(K_w, N)$ is zero. The long exact cohomology sequence associated to the exact triangle (3.1) therefore induces an identification of $H^0_c(\mathcal{O}_{K,S}, N)$ with the kernel of the natural localisation homomorphism $H^0(\mathcal{O}_{K,S}, N) \to P^0(\mathcal{O}_{K,S}, N)$.

Now $H^0(\mathcal{O}_{K,S}, N)$ is canonically isomorphic to $N^{G_{K,S}}$ and $P^0(\mathcal{O}_{K,S}, N)$ is canonically isomorphic to $\bigoplus_{w \in S(K)} N^{G_w}$. Further, since G_w is a subgroup of G_K and G_K acts on N through its quotient $G_{K,S}$, with respect to these isomorphisms the localisation homomorphism $H^0(\mathcal{O}_{K,S}, N) \to P^0(\mathcal{O}_{K,S}, N)$ identifies with the (injective) diagonal map $N^{G_{K,S}} \hookrightarrow \bigoplus_{w \in S(K)} N^{G_w}$. Hence $H^0_c(\mathcal{O}_{K,S}, N) = 0$, as claimed. □

3.1.3 Finite Support Cohomology

The finite support cohomology was introduced by Bloch and Kato in [6] and is described in detail by Burns and Flach in [11, §3.2]. For the convenience of the reader, we give only a brief review here referring to the indicated references for further details.

Definition 3.1.7. Let r be an integer.

- For each place w of K the *finite support cohomology complex* for the G_w-module $\mathbb{Q}_p(r)$ is defined by setting

$$R\Gamma_f(K_w, \mathbb{Q}_p(r)) := \begin{cases} R\Gamma(K_w, \mathbb{Q}_p(r)), & \text{if } w \mid \infty, \\ \left[\mathbb{Q}_p(r)^{I_w} \xrightarrow{1-f_w^{-1}} \mathbb{Q}_p(r)^{I_w}\right], & \text{if } w \nmid \infty \text{ and } w \nmid p, \\ (B^\bullet \otimes \mathbb{Q}_p(r))^{G_w}, & \text{if } w \mid p \end{cases}$$

  where $B^\bullet$ is the canonical complex (concentrated in degrees 0 and 1) that is defined in [6, Prop. 1.17].

- For each place w of K we write $R\Gamma_{/f}(K_w, \mathbb{Q}_p(r))$ for a complex of $\mathbb{Q}_p[G]$-modules which fits into an exact triangle

$$R\Gamma_f(K_w, \mathbb{Q}_p(r)) \to R\Gamma(K_w, \mathbb{Q}_p(r)) \to R\Gamma_{/f}(K_w, \mathbb{Q}_p(r)) \to, \qquad (3.2)$$

where the first arrow is a canonical morphism constructed in [11, §3.2]. We note that if w is archimedean, then $R\Gamma_{/f}(K_w, \mathbb{Q}_p(r))$ is acyclic.

- The finite support cohomology complex for the $G_{K,S}$-module $\mathbb{Q}_p(r)$ is defined so as to lie in an exact triangle

$$R\Gamma_f(K, \mathbb{Q}_p(r)) \to R\Gamma(\mathcal{O}_{K,S}, \mathbb{Q}_p(r)) \to \bigoplus_{\substack{w \in S(K) \\ w \nmid \infty}} R\Gamma_{/f}(K_w, \mathbb{Q}_p(r)) \to, \qquad (3.3)$$

  where the second arrow is the morphism induced by the composite of the natural localisation morphism $R\Gamma(\mathcal{O}_{K,S}, \mathbb{Q}_p(r)) \to \bigoplus_{w \in S(K)} R\Gamma(K_w, \mathbb{Q}_p(r))$ with the second arrow of the exact triangle (3.2).

In general the finite support cohomology complex is not defined for the module $\mathbb{Z}_p(r)$ (rather than the space $\mathbb{Q}_p(r)$). Nevertheless, for each place w of K it is always possible to define the finite support cohomology group in degree 1 for the G_w module $\mathbb{Z}_p(r)$ as follows.

Definition 3.1.8. Given any non-archimedean place w of K one has $H^0_{/f}(K_w, \mathbb{Q}_p(r)) = 0$ and so the triangle (3.2) induces a canonical injection

$$H^1_f(K_w, \mathbb{Q}_p(r)) \hookrightarrow H^1(K_w, \mathbb{Q}_p(r)). \qquad (3.4)$$

We use this map to identify its domain with its image and then make the following definitions.

- For each non-archimedean place w of K define $H^1_f(K_w, \mathbb{Z}_p(r))$ to be the full pre-image of $H^1_f(K_w, \mathbb{Q}_p(r))$ under the natural map $H^1(K_w, \mathbb{Z}_p(r)) \to H^1(K_w, \mathbb{Q}_p(r))$.

- For each non-archimedean place w of K define $H^1_f(K_w, \mathbb{Q}_p/\mathbb{Z}_p(r))$ to be the image of $H^1_f(K_w, \mathbb{Q}_p(r))$ under the natural homomorphism $H^1(K_w, \mathbb{Q}_p(r)) \to H^1(K_w, \mathbb{Q}_p(r)/\mathbb{Z}_p(r))$.

- For each archimedean place w of K define both $H^1_f(K_w, \mathbb{Z}_p(r))$ and $H^1_f(K_w, \mathbb{Q}_p/\mathbb{Z}_p(r))$ to be zero.

- Given any place w of K we set $H^1_{/f}(K_w, \mathbb{Z}_p(r)) := H^1(K_w, \mathbb{Z}_p(r))/H^1_f(K_w, \mathbb{Z}_p(r))$ and $H^1_{/f}(K_w, \mathbb{Q}_p/\mathbb{Z}_p(r)) := H^1(K_w, \mathbb{Q}_p/\mathbb{Z}_p(r))/H^1_f(K_w, \mathbb{Q}_p/\mathbb{Z}_p(r)))$.

- Define the module $H^1_f(K, \mathbb{Z}_p(r))$ to be the kernel of the natural composite homomorphism $H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \twoheadrightarrow P^1_{/f}(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$.

Remark 3.1.9. Let r be an integer strictly greater than one. Then in [6, Ex. 3.9] Bloch and Kato show that for any place w of K the injection (3.4) is an isomorphism. In this case therefore one has:

- $H^1_f(K_w, \mathbb{Z}_p(r)) = H^1(K_w, \mathbb{Z}_p(r))$ for each place w of K.

- $H^1_{/f}(K_w, \mathbb{Z}_p(r)) = 0$ for each place w of K.

- $H^1_f(K, \mathbb{Z}_p(r)) = H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$.

3.2 Tate-Shafarevic Groups of Tate Motives

We fix a finite abelian extension of number fields K/k with $G := \mathrm{Gal}(K/k)$ and a finite set of places S of k containing all archimedean places, all places above p and all that ramify in K/k.

3.2.1 Definitions and Basic Properties

We recall now the two types of Tate-Shafarevic groups that are to be used in this thesis. The first is the Tate-Shafarevic group in the sense of Neukirch, Schmidt and Wingberg.

Definition 3.2.1. Given any positive integer n and any topological $G_{K,S}$-module N the 'Tate-Shaferevic group in degree n' of $\mathcal{O}_{K,S}$ and N is denoted by $\text{Ш}^n(\mathcal{O}_{K,S}, N)$ and is defined in [45, Defn. 8.6.2] to be the kernel of the natural localisation homomorphism $H^n(\mathcal{O}_{K,S}, N) \to P^n(\mathcal{O}_{K,S}, N)$.

The second is the Tate-Shafarevic group in the sense of Bloch and Kato.

Definition 3.2.2. The *Bloch-Kato-Selmer group* $\mathrm{Sel}(\mathbb{Z}_p(r)_K)$ of the module $\mathbb{Z}_p(r)_K$ is defined in [6] to be equal to the kernel of the composite homomorphism

$$H^1(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r)) \to P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r)) \twoheadrightarrow P^1_{/f}(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r)).$$

We recall also that the definition in loc. cit. of the *Bloch-Kato Tate-Shafarevic group* $\text{Ш}(\mathbb{Z}_p(r)_K)$ of the module $\mathbb{Z}_p(r)_K$ ensures that it is equal to $\mathrm{Sel}(\mathbb{Z}_p(r)_K)_{\mathrm{cotor}}$ (cf. [28, Chap. II, 5.3.4-5.3.5]). We note that both $\mathrm{Sel}(\mathbb{Z}_p(r)_K)$ and $\text{Ш}(\mathbb{Z}_p(r)_K)$ are naturally $\mathbb{Z}_p[G]$-modules.

We shall make much use of the following result of Flach (from [25, Thm. 1]).

Proposition 3.2.3. *For each integer r there is a canonical isomorphism of $\mathbb{Z}_p[G]$-modules*

$$\text{Ш}(\mathbb{Z}_p(r)_K) \cong \text{Ш}(\mathbb{Z}_p(1-r)_K)^\vee.$$

Throughout this thesis we will use both types of Tate-Shafarevic groups and so it is convenient to describe the relationship between them.

Lemma 3.2.4.

(i.) For any integer r there exists a natural surjection of $\mathbb{Z}_p[G]$-modules

$$H^2_c(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \twoheadrightarrow \mathrm{Sel}(\mathbb{Z}_p(1-r)_K)^\vee.$$

(ii.) If r is strictly greater than one then there are canonical isomorphisms of $\mathbb{Z}_p[G]$-modules

$$\mathrm{Sel}(\mathbb{Z}_p(1-r)_K)^\vee \cong Ш^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \cong Ш(\mathbb{Z}_p(1-r)_K)^\vee \cong Ш(\mathbb{Z}_p(r)_K).$$

Proof. From the very definition of $H^1_f(K,\mathbb{Z}_p(r))$ there exists a natural exact sequence $0 \to H^1_f(K,\mathbb{Z}_p(r)) \xrightarrow{\iota} H^1(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \to P^1_{/f}(\mathcal{O}_{K,S},\mathbb{Z}_p(r))$ and hence also a canonical homomorphism $\alpha : H^1_f(K,\mathbb{Z}_p(r)) \to P^1_f(\mathcal{O}_{K,S},\mathbb{Z}_p(r))$. Combining this with the long exact cohomology sequence associated to the exact triangle (3.1) one obtains a commutative diagram

$$\begin{tikzcd}
H^1(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \arrow[r] & P^1(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \arrow[r] & H^2_c(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \\
H^1_f(K,\mathbb{Z}_p(r)) \arrow[u, hook, "\iota"] \arrow[r, "\alpha"] & P^1_f(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \arrow[u, hook] &
\end{tikzcd}$$

in which the top row is exact. Hence there exists an exact sequence of the form

$$H^1_f(K,\mathbb{Z}_p(r)) \xrightarrow{\alpha} P^1_f(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \xrightarrow{\alpha'} H^2_c(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \to \mathrm{Cok}(\alpha') \to 0. \qquad (3.5)$$

Now for each place w in $S(K)$ the local duality theorem of [45, Thm. 7.2.1] gives a canonical isomorphism $H^1(K_w,\mathbb{Z}_p(r))^\vee \cong H^1(K_w,\mathbb{Q}_p/\mathbb{Z}_p(1-r))$ which in turn induces a canonical isomorphism between $H^1_f(K_w,\mathbb{Z}_p(r))^\vee$ and $H^1_{/f}(K_w,\mathbb{Q}_p/\mathbb{Z}_p(1-r))$. By combining the Pontryagin dual of (3.5) with the Artin-Verdier duality theorem (cf. [43, Chap. II, Cor. 3.2]) we therefore obtain a natural exact sequence

$$0 \to \mathrm{Cok}(\alpha')^{\vee} \to H^1(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r)) \to P^1_{/f}(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r)).$$

But $\mathrm{Sel}(\mathbb{Z}_p(1-r)_K)$ is defined to be the kernel of the last map and so $\mathrm{Cok}(\alpha')$ is isomorphic to $\mathrm{Sel}(\mathbb{Z}_p(1-r)_K)^{\vee}$. The sequence (3.5) therefore induces a short exact sequence

$$0 \to \mathrm{Cok}(\alpha) \to H^2_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to \mathrm{Sel}(\mathbb{Z}_p(1-r)_K)^{\vee} \to 0, \tag{3.6}$$

as required by claim (i).

To prove claim (ii) we assume that $r > 1$. Then Remark 3.1.9 implies that in this case the canonical injections $H^1_f(K, \mathbb{Z}_p(r)) \hookrightarrow H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ and $P^1_f(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \hookrightarrow P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ are both bijective. Hence (3.6) combines with the long exact sequence of the distinguished triangle (3.1) to imply that $\mathrm{Sel}(\mathbb{Z}_p(1-r)_K))^{\vee}$ is isomorphic to the image of the natural homomorphism $H^2_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to H^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ and therefore also to $Ш^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$. Now by Lemma 3.2.5 (below) the group $H^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$, and hence also its subgroup $Ш^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \cong \mathrm{Sel}(\mathbb{Z}_p(1-r)_K))^{\vee}$, is finite. This implies that $\mathrm{Sel}(\mathbb{Z}_p(1-r)_K)^{\vee}$ is equal to $(\mathrm{Sel}(\mathbb{Z}_p(1-r)_K)_{\mathrm{cotor}})^{\vee} = Ш(\mathbb{Z}_p(1-r)_K)^{\vee}$ and also gives a natural isomorphism of the form $Ш^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \cong Ш(\mathbb{Z}_p(1-r)_K)^{\vee}$. The final isomorphism in claim (ii) follows directly from Proposition 3.2.3. This completes the proof of claim (ii). $\square$

Before proving the next result we recall that for any number field K, any odd prime p, any finite set of places Σ of K containing all archimedean places and all places lying above p, and any integers n and r with $n \in \{1, 2\}$ and $r \geq n$ Soulé [57] and Dwyer-Friedlander [22] have defined canonical "Chern-class" homomorphisms of the form

$$\mathrm{ch}^r_{K,\Sigma,p,n} : K_{2r-n}\left(\mathcal{O}_{K,\Sigma}\right) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^n\left(\mathcal{O}_{K,\Sigma}, \mathbb{Z}_p(r)\right). \tag{3.7}$$

We further recall that these homomorphisms have been shown to to be surjective and to have finite kernel and in particular therefore that the induced homomorphisms $\mathrm{ch}^r_{K,\Sigma,p,n} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ are bijective.

Lemma 3.2.5. *If $r > 1$, then the group $H^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is finite.*

Proof. If $r > 1$, then $K_{2r-2}(\mathcal{O}_{K,S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is finite by work of Borel [7]. The claimed result therefore follows immediately from the bijectivity of $\mathrm{ch}^r_{K,S,p,2} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the fact that $H^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is a finitely generated $\mathbb{Z}_p$-module. □

3.2.2 The Quillen-Lichtenbaum Conjecture and Wild Kernels

If one wants to describe explicitly the groups $Ш(\mathbb{Z}_p(r)_K)$, then Flach's duality result (Proposition 3.2.3) allows a reduction to the case that r is strictly positive. Further, in [25, Ex. p.122-123] Flach has shown that there is a canonical isomorphism of $\mathbb{Z}_p[G]$-modules of the form

$$\mathrm{Cl}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong Ш(\mathbb{Z}_p(1)_K).$$

Given any integer r with $r > 1$ it is therefore natural to ask whether there is an analogous description of $Ш(\mathbb{Z}_p(r)_K)$. To investigate this question we first need to determine an appropriate analogue of $\mathrm{Cl}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. In this regard we recall that for each $r > 1$ Banaszak has defined in [3, §I, Defn. 1] a natural subgroup $K^w_{2r-2}(\mathcal{O}_K)_p$ of $K_{2r-2}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ which is called the 'p-adic wild kernel'. It is defined to be the kernel of a natural homomorphism

$$K_{2r-2}(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \bigoplus_w (\mathbb{Q}_p/\mathbb{Z}_p(r-1))^{G_w}$$

where w runs over all places of K. To explain why we feel that $K^w_{2r-2}(\mathcal{O}_K)_p$ is the natural analogue of $\mathrm{Cl}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ we first recall the following well-known conjecture of Quillen and Lichtenbaum.

Conjecture 3.2.6 (Quillen-Lichtenbaum)**.** *(cf. [35, Introduction])*

For any odd prime p, any number field K and any finite set of places Σ of K containing all archimedean places of K and all places lying above p, the chern class maps $\mathrm{ch}^r_{K,\Sigma,p,n}$ are bijective for both $n \in \{1, 2\}$ and all $r > 1$.

Lemma 3.2.7. *Fix an integer r with $r > 1$. If K validates Conjecture 3.2.6 for p and r and with $n = 2$, then $\mathrm{ch}^r_{K,S,2}$ induces a canonical isomorphism*

$$K^w_{2r-2}(\mathcal{O}_K)_p \cong \text{Ш}(\mathbb{Z}_p(r)_K).$$

Proof. For any place $w \in S(K)$, local duality (cf. [45, Thm. 7.2.1]) gives a natural isomorphism of the form $H^2(K_w, \mathbb{Z}_p(r)) \cong ((\mathbb{Q}_p/\mathbb{Z}_p(1-r))^{G_w})^\vee$. Since $\chi^{r-1}_{\mathrm{cyclo}}(g^{-1}) = -\chi^{1-r}_{\mathrm{cyclo}}(g)$ for each $g \in G_w$ there is also a canonical isomorphism $((\mathbb{Q}_p/\mathbb{Z}_p(1-r))^{G_w})^\vee \cong (\mathbb{Q}_p/\mathbb{Z}_p(r-1))^{G_w}$. Taking this into account one finds that if $\mathrm{ch}^r_{K,S,2}$ is bijective, then the localisation homomorphism $\rho^2_r : H^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to P^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ corresponds to the composite map

$$\begin{aligned}\kappa_p : K_{2r-2}(\mathcal{O}_{K,S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\lambda'_p} K_{2r-2}(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\lambda_p} \\ \bigoplus_{w\in\Sigma(K)} \mathbb{Q}_p/\mathbb{Z}_p(r-1)^{G_w} \xrightarrow{\pi_{S(K)}} \bigoplus_{w\in S(K)} \mathbb{Q}_p/\mathbb{Z}_p(r-1)^{G_w}, \quad (3.8)\end{aligned}$$

where λ'_p is the localisation map in K-theory, λ_p the map in [3, Th. 2], $\Sigma(K)$ the set of all places of K and for any subset S' of $\Sigma(K)$ we write $\pi_{S'}$ for the projection $\bigoplus_{w\in\Sigma(K)} \mathbb{Q}_p/\mathbb{Z}_p(r-1)^{G_w} \to \bigoplus_{w\in S'} \mathbb{Q}_p/\mathbb{Z}_p(r-1)^{G_w}$. But from the exact localisation sequence of K-theory one knows that λ'_p is injective and has image equal to $\ker(\pi_{\Sigma(K)\setminus S(K)} \circ \lambda_p)$ (cf. [3, §III.1]) and this implies that λ'_p induces an isomorphism $Ш^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) := \ker(\kappa_p) = \ker(\lambda_p \circ \lambda'_p) \cong \ker(\lambda_p) =: K^w_{2r-2}(\mathcal{O}_K)_p$. The claimed isomorphism $K^w_{2r-2}(\mathcal{O}_K)_p \cong Ш(\mathbb{Z}_p(r)_K)$ therefore follows from Lemma 3.2.4(ii). $\square$

Remark 3.2.8.

(i) When $r = 2$ the Quillen-Lichtenbaum conjecture (Conjecture 3.2.6) is known to be true for $n = 2$ by work of Tate [61] and for $n = 1$ by work of Levine [40] and Merkuriev and Suslin [42].

(ii) Suslin and Voevodsky [59] have proved that the Quillen-Lichtenbaum conjecture is implied by the Bloch-Kato conjecture described in [29]. We understand that this conjecture of Bloch and Kato (and hence also Conjecture 3.2.6) has recently been proved unconditionally in unpublished work of M. Rost and C. Weibel.

3.2.3 The Conjectures of Leopoldt and Schneider

We first recall (from, for example, [45, Conj. 10.3.5]) that Leopoldt's conjecture for K at p asserts that the p-adic regulator rank of K (as defined in [45, Defn. 10.3.3]) is equal to $r_1 + r_2 - 1$, where r_1 is the number of real places of K and r_2 the number of complex places of K. We further recall that $Ш^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p) = H^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p)$ and that it is known that K validates Leopoldt's conjecture at p if and only if the group $H^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p)$ vanishes (cf. [45, Thm. 10.3.6, Rem. p.550]). Partly motivated by these observations, in [50] Schneider formulated the following generalisation of Leopoldt's conjecture.

Conjecture 3.2.9 (Schneider's Conjecture for K at r and p)**.** *If r is any non-zero integer then the cohomology group $H^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$ is finite.*

The following observations regarding Schneider's Conjecture will be useful in the sequel.

Lemma 3.2.10.

(i) If r is any strictly negative integer and p any prime, then Schneider's conjecture for K at r and p is valid.

(ii) K validates Schneider's conjecture at a non-zero integer r and an odd prime p if and only if the group $\text{Ш}^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ vanishes.

(iii) If K is a CM abelian extension of a totally real field k and r is strictly greater than one, then the group $\text{Ш}^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$ vanishes.

(iv) If L/k is a finite abelian extension of totally real fields and r is a strictly positive even integer, then Schneider's conjecture holds for L at r and p.

(v) If K is a CM abelian extension of a totally real field k, K^+ is the maximal totally real subfield of K and r is a positive odd integer, then Schneider's conjecture holds for K at r and p if and only if it holds for K^+ at r and p.

Proof. Claim (i) is proved by Soulé in [57] (see, for example, [3, Lem. 1]).

To prove the remaining claims we note first that the group $\text{Ш}^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is torsion-free, and hence vanishes if and only if it is finite. To show this we set $\Sigma := \{v \in S \mid v \nmid \infty\}$ and observe that since $r \neq 0$ the groups $H^0(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ and $P^0(\mathcal{O}_{K,\Sigma}, \mathbb{Q}_p(r))$ both vanish and hence that the long exact cohomology sequence induced by the short exact sequence $0 \to \mathbb{Z}_p(r) \to \mathbb{Q}_p(r) \to \mathbb{Q}_p/\mathbb{Z}_p(r) \to 0$ induces canonical isomorphisms $H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))_{\text{tor}} \cong H^0(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r))$ and also

$P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))_{\mathrm{tor}} = P^1(\mathcal{O}_{K,\Sigma}, \mathbb{Z}_p(r))_{\mathrm{tor}} \cong P^0(\mathcal{O}_{K,\Sigma}, \mathbb{Q}_p/\mathbb{Z}_p(r)) \subseteq P^0(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r))$. There is therefore a canonical injection $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))_{\mathrm{tor}} \hookrightarrow H^0_c(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r))$. But $H^0_c(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r))$ vanishes (by Lemma 3.1.6) and so the group $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is torsion-free, as required.

Next we note that Poitou-Tate duality implies that $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is isomorphic to $Ш^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee$ (cf. [45, Thm. 8.6.8]). In addition, for each place w of K the group $H^2(K_w, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$ vanishes (cf. [45, Cor. 7.2.6]) and so $Ш^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r)) = H^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$. It follows that the group $H^2(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$ is finite if and only if $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is finite which, by the observation made above, is equivalent to the vanishing of $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$. This proves claim (ii).

We assume now that $r > 1$. Then the chern class map (3.7) for r and $n = 1$ induces an isomorphism of $\mathbb{Q}_p[G]$-modules $H^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r)) \cong K_{2r-1}(\mathcal{O}_{K,S}) \otimes_{\mathbb{Z}} \mathbb{Q}_p$. Further, since $r > 1$, there is also a canonical isomorphism of groups $K_{2r-1}(\mathcal{O}_{K,S}) \cong K_{2r-1}(\mathcal{O}_K)$ (see, for example, [17, Prop. 5.7]). Hence by Lemma 3.2.12 (below) we see that the group $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$ is finite and hence, since it is also torsion-free, vanishes. This proves claim (iii).

For claim (iv) we fix a quadratic imaginary extension E of k that is unramified outside the given set S. Then the field $K := EL$ is a CM field with maximal real subfield L. Further, since r is even, one has $e_r^- = e^+$ (by Remark 2.2.2) and so the canonical descent isomorphisms of [15, §3.1] give isomorphisms $H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^- \cong H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ and $P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^- \cong P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ and hence also a canonical isomorphism $Ш^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \cong Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$. The latter group vanishes by claim (iii) and so the validity of Schneider's conjecture for L at p and r follows from claim (ii). This proves claim (iv).

To prove claim (v) we assume that K, K^+ and r are as given and use the direct sum decomposition $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) = Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^- \oplus Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^+$ and the isomorphism $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^+ = Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e^+ \cong Ш^1(\mathcal{O}_{K^+,S}, \mathbb{Z}_p(r))$ that is established just as in the last paragraph (using the fact that r is now odd). By combining these facts with claims (i) and (ii) it follows that Schneider's conjecture is valid for K at r and p if and only if the group $Ш^1(\mathcal{O}_{K^+,S}, \mathbb{Z}_p(r))$ vanishes, which by claim (ii) is in turn true if and only if Schneider's conjecture is valid for K^+ at p and r. This proves claim (v). □

Remark 3.2.11. Following upon Lemma 3.2.10(i) and (iii) it is important to note that if $r \geq 1$, then Schneider's conjecture for K at r and p is still very much open. Indeed, our earlier observations make it clear that Schneider's conjecture for K at $r = 1$ and p is equivalent to Leopoldt's conjecture for K at p.

Lemma 3.2.12. *Let r be an integer strictly greater than one.*

(i) The Beilinson regulator induces a canonical isomorphism of $\mathbb{R}[G]$-modules

$$R_{K,r} : K_{2r-1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{R} \cong Y_{K,r-1}^+ \otimes_{\mathbb{Z}} \mathbb{R}. \tag{3.9}$$

(ii) There exists an isomorphism of $\mathbb{Q}[G]$-modules

$$K_{2r-1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q} \cong Y_{K,r-1}^+ \otimes_{\mathbb{Z}} \mathbb{Q}.$$

(iii) If K is a CM abelian extension of a totally real field k, then $K_{2r-1}(\mathcal{O}_K)e_r^-$ is finite.

Proof. Claim (i) is proved in [9, §10] and claim (ii) follows directly upon combining the isomorphism of claim (i) with Deuring's theorem (cf. [19, Thm. 29.7]).

Claim (ii) implies that claim (iii) is valid if e_r^- annihilates $Y_{K,r-1}^+ \otimes_{\mathbb{Z}} \mathbb{Q}$. But G, resp. $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, acts on $Y_{K,r-1} = \bigoplus_{K \hookrightarrow \mathbb{C}} (2\pi i)^{r-1}\mathbb{Z}$ via pre-composition with the indexing maps, resp. diagonally by the natural action on the coefficients and by post-composition with the indexing maps, and hence the unique non-trivial element τ of $\mathrm{Gal}(K/K^+)$ acts on $Y_{K,r-1}^+$ like multiplication by $(-1)^{r-1}$. It is therefore clear that the idempotent $e_r^- = \frac{1+(-1)^r\tau}{2} = \frac{1-(-1)^{r-1}\tau}{2}$ annihilates $Y_{K,r-1}^+ \otimes_{\mathbb{Z}} \mathbb{Q}$. $\square$

Chapter 4

The Generalised Iwasawa Main Conjecture and Gras-Oriat-type Annihilators

Let k be a totally real number field and K a finite abelian extension of k that is either totally real or a CM field. Let S denote the (finite) set of places of k comprising all which ramify in K/k, all archimedean places and all places which lie above p.

If K is totally real, then in this chapter we shall use the leading term at $s = 1$ of the equivariant S-truncated p-adic L-function of K/k to construct elements of $\mathbb{Z}_p[\mathrm{Gal}(K/k)]$ that annihilate the ideal class group of K. This result is a natural strengthening of results of Oriat [47, Thms. A & B] which were themselves an extension of previous work of Gras. If K is a CM field, then for each integer r strictly greater than one we shall also prove that the values at both $s = r$ and $s = 1-r$ of the "twisted" equivariant S-truncated p-adic L-function of K/k can be used to construct elements of $\mathbb{Z}_p[\mathrm{Gal}(K/k)]$ that annihilate the Tate-Shafarevic group $\text{Ш}(\mathbb{Z}_p(r)_K)$.

We prove these results by combining a special case of the generalised main conjecture of Iwasawa theory (which we prove in §4.2) together with a natural generalisation of an algebraic result of Snaith (which is joint work with David Burns and is both stated and proved in §4.3) and an explicit description of the étale cohomology groups $H^n_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ for each non-negative integer n and non-zero integer r.

4.1 Statement of the Main Results

Since p is fixed, for any finite group H we write I_H for the kernel of the $\mathbb{Z}_p$-module homomorphism $\epsilon_H : \mathbb{Z}_p[H] \to \mathbb{Z}_p$ which maps each element of H to 1. We also write K^∞ for the cyclotomic $\mathbb{Z}_p$-extension of K and abbreviate the Galois groups $\mathcal{G}_{K,S_{K/k}}$ and $\mathcal{H}_{K,S_{K/k}}$ introduced in §2.2.1 to $\mathcal{G}_K$ and $\mathcal{H}_K$ respectively. We shall use both the equivariant S-truncated p-adic L-functions $L_{p,S}(s)$ introduced in Definition 2.2.7 and the twisted version $\mathfrak{L}_{p,S}(\cdot)$ of these functions introduced in Definition 2.2.8.

4.1.1 The case $r = 1$

The following result will be proved in §4.5.

Theorem 4.1.1. *Let L/k be a finite abelian extension of totally real number fields. Let $H(L)$ denote the Hilbert class-field of L and write $\mathrm{Cl}_{(\infty)}(\mathcal{O}_L)$ for the subgroup of $\mathrm{Cl}(\mathcal{O}_L)$ which corresponds via class field theory to $\mathrm{Gal}(H(L)/H(L) \cap L^\infty)$. Assume that Leopoldt's conjecture holds for L at p and that L is the maximal totally real subfield of an abelian CM extension K of k which is unramified outside of $S = S_{L/k}$ and such that $\mu(K, p)$ vanishes. Then*

$$L^*_{p,S}(1)I_G^2 \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathcal{H}_L) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{(\infty)}(\mathcal{O}_L)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \tag{4.1}$$

and

$$L^*_{p,S}(1)I_G^3 \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_L)) \otimes_{\mathbb{Z}} \mathbb{Z}_p. \tag{4.2}$$

If, in addition, the Sylow p-subgroup of G is cyclic, then

$$L^*_{p,S}(1)I_G = \mathrm{Fitt}_{\mathbb{Z}_p[G]}(\mathcal{H}_L) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{(\infty)}(\mathcal{O}_L)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \tag{4.3}$$

and

$$L^*_{p,S}(1)I_G^2 \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_L)) \otimes_{\mathbb{Z}} \mathbb{Z}_p. \tag{4.4}$$

Remark 4.1.2.

(i) Let L/k be a finite abelian extension of totally real fields for which there exists a quadratic imaginary extension E of k such that E/k is unramified outside $S_{L/k}$ and $\mu(EL, p) = 0$. Then L is the maximal totally real subfield of the CM field $K := EL$ and the abelian extension K/k is unramified outside of $S_{L/k}$. In particular, if Leopoldt's conjecture holds for L at p then we may apply Theorem 4.1.1 to the extension L/k.

(ii) If in the setting of Remark (i) we also assume that $L/\mathbb{Q}$ is abelian, then $K/\mathbb{Q}$ is also abelian so that both $\mu(K, p) = 0$ (by Ferrero-Washington [24]) and L validates Leopoldt's conjecture at p (by Brumer [8]). Theorem 4.1.1 is therefore unconditional in this case.

(iii) If $H(L) \cap L^\infty = L$ (which is true, for example, if L^∞/L is totally ramified), then $\mathrm{Cl}_{(\infty)}(\mathcal{O}_L) = \mathrm{Cl}(\mathcal{O}_L)$ and so the inclusions (4.1) and (4.3) are strictly stronger than those of (4.2) and (4.4).

The following consequence of Theorem 4.1.1 is a strengthening of the main results of Oriat in [47].

Corollary 4.1.3. *Let L be a non-trivial totally real cyclic extension of $\mathbb{Q}$, ζ a primitive $|G|^{th}$ root of unity and $\chi : G \to \overline{\mathbb{Q}}^\times$ a faithful character of G. Let $\widetilde{\chi}$ be the natural extension $\mathbb{C}_p[G] \to \mathbb{C}_p$ of χ. Then*

$$(1-\zeta)L_p(\chi, 1) \in \mathrm{Fitt}_{\mathbb{Z}_p[\zeta]}(\mathcal{H}_L \otimes_{\mathbb{Z}_p[G],\widetilde{\chi}} \mathbb{Z}_p[\zeta]) \tag{4.5}$$

and, if $|G|$ is not a power of p, also

$$L_p(\chi, 1) \in \mathrm{Fitt}_{\mathbb{Z}_p[\zeta]}(\mathcal{H}_L \otimes_{\mathbb{Z}_p[G],\widetilde{\chi}} \mathbb{Z}_p[\zeta]).$$

Proof. Since G is cyclic Remark 4.1.2(ii) implies that the equality $L^*_{p,S}(1)I_G = \mathrm{Fitt}_{\mathbb{Z}_p[G]}(\mathcal{H}_L)$ is valid unconditionally. Next we note that since G is not trivial the (faithful) character χ is not the trivial character and so the leading term of the function $L_{p,S}(\chi, s)$ at $s = 1$ is equal to its value $L_{p,S}(\chi, 1)$ at $s = 1$: the image of $L^*_{p,S}(1)$ under $\tilde{\chi}$ is therefore equal to $L_{p,S}(\chi, 1)$.

We now let g be the generator of G for which $\chi(g) = \zeta$. Then one has both $I_G = \mathbb{Z}_p[G] \cdot (1 - g)$ and $\widetilde{\chi}(1 - g) = 1 - \zeta$. By projecting the equality of (4.3) under $\tilde{\chi}$ we therefore obtain a containment

$$(1 - \zeta)L_{p,S}(\chi, 1) \in \mathrm{Fitt}_{\mathbb{Z}_p[\zeta]}\left(\mathcal{H}_L \otimes_{\mathbb{Z}_p[G],\widetilde{\chi}} \mathbb{Z}_p[\zeta]\right). \tag{4.6}$$

To deduce (4.5) from this it only remains to show that $L_{p,S}(\chi, 1) = L_p(\chi, 1)$. But if q is any prime which ramifies in $L/\mathbb{Q}$, then for each q-adic place w of L the inertia group I_w is non-trivial and so the space $V_\chi^{I_w}$ vanishes (since χ is faithful) and hence the associated Euler factor $\det(1 - f_w q^{-1} \mid V_\chi^{I_w})$ is equal to 1.

Finally we note that if $|G|$ is not a power of p, then the element $1 - \zeta$ is a (cyclotomic) unit of $\mathbb{Z}_p[\zeta]$ and so (4.6) implies that the element $L_p(\chi, 1)$ belongs to $\mathrm{Fitt}_{\mathbb{Z}_p[\zeta]}\left(\mathcal{H}_L \otimes_{\mathbb{Z}_p[G],\widetilde{\chi}} \mathbb{Z}_p[\zeta]\right)$, as required. □

Remark 4.1.4. With L as in Corollary 4.1.3 we write $M_p(L)$ for the maximal abelian pro-p extension of L that is unramified outside of the places lying above p. Then $L^\infty \subset M_p(L) \subset M^p_{S_{L/\mathbb{Q}}}(L)$ and so $\mathrm{Gal}(M_p(L)/L^\infty)$ is a quotient of $\mathcal{H}_L$. Given this observation, it is immediately clear that the result of Corollary 4.1.3 is stronger than the main results (Theorems A and B) of Oriat in [47].

4.1.2 The case $r > 1$

In the following result we use the "twisted" equivariant p-adic L-functions $\mathfrak{L}_{p,S}(\cdot)$ introduced in Definition 2.2.8. This result will be proved in §4.6.

Theorem 4.1.5. *Let K be a CM abelian extension of a totally real field k and set $G := \mathrm{Gal}(K/k)$. Let r be an integer strictly greater than one for which Schneider's conjecture (Conjecture 3.2.9) holds for K at r and p. Assume also that K contains a primitive p^{th} root of unity and that the μ-invariant $\mu(K,p)$ vanishes.*

(i) One has

$$\mathfrak{L}_{p,S}(r)\,\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_K}) + \mathfrak{L}_{p,S}(1-r)\,\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(-r)_{G_K}) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}\left(\text{Ш}(\mathbb{Z}_p(r)_K)\right).$$

(ii) If the Sylow p-subgroup of G is cyclic, then also

$$\mathfrak{L}_{p,S}(r)\,\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_K}) \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]}\left(\text{Ш}(\mathbb{Z}_p(r)_K)\right).$$

(iii) If the Quillen-Lichtenbaum conjecture (Conjecture 3.2.6) holds for r and $n = 2$, then we may replace $\text{Ш}(\mathbb{Z}_p(r)_K)$ in claims (i) *and* (ii) *with the p-adic wild kernel $K^w_{2r-2}(\mathcal{O}_K)_p$.*

Remark 4.1.6. If K is abelian over $\mathbb{Q}$, then $\mu(K,p)$ vanishes (by Ferrero-Washington [24]) and so Theorem 4.1.5 is conditional only upon the validity of Schneider's conjecture for K at r and p.

4.2 The Generalised Iwasawa Main Conjecture

The key to our proof of Theorems 4.1.1 and 4.1.5 is an explicit formula for the $\mathbb{Z}_p[G]$-ideal generated by the leading term at an integer r of the equivariant p-adic L-function in terms of the determinant of a canonical perfect complex. This result is stated below as Theorem 4.2.1 and will be proved in this section. In fact we shall explain in Appendix B that, modulo the validity of Serre's p-adic Stark Conjecture at $s = 1$, resp. of a natural generalisation of this conjecture dealing with leading terms at integers strictly greater than one, the result of Theorem 4.2.1 for $r = 1$, resp. $r > 1$, is equivalent to a special case of the equivariant Tamagawa number conjecture of Burns and Flach. Since G is abelian, this result is therefore also a natural refinement of a special case of the Generalised Iwasawa Main Conjecture formulated by Kato in [36].

4.2.1 Statement of the leading term formula

Theorem 4.2.1. *Let K be a CM abelian extension of a totally real field k. Fix an integer r, denote by $e_r^{(0)}$ the sum over all primitive idempotents of $\mathbb{Q}_p[G]$ which annihilate the space $H^2_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ and set $\mathfrak{A}_r^{(0)} := \mathbb{Z}_p[G]e_r^{(0)}$.*

(i) One has $e_r^{(0)} e_r^+ = e_r^{(0)}$.

(ii) If $\mu(K, p)$ vanishes, then there is an equality of graded invertible $\mathfrak{A}_1^{(0)}(1 - e_G)$-modules

$$\left[R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(1)) \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}_1^{(0)}(1 - e_G)\right]^{-1}_{\mathfrak{A}_1^{(0)}(1-e_G)} = ((1 - e_G)\mathfrak{A}_1^{(0)} L^*_{p,S}(1), 0).$$

(iii) If $r \neq 1$, $\mu(K, p)$ vanishes and K contains a primitive p^{th}-root of unity then

there exists an equality of graded invertible $\mathfrak{A}_r^{(0)}$-modules

$$\big[R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^{(0)}\big]^{-1}_{\mathfrak{A}_r^{(0)}} = (\mathfrak{A}_r^{(0)}\mathfrak{L}_{p,S}(r),0).$$

4.2.2 The proof of Theorem 4.2.1(i)

It is well known that the cohomological dimension of the groups $G_{K,S}$ and G_w for each place w in $S(K)$ is equal to 2 (cf. [45, Thms. 7.1.8 & 10.9.3]). From the long exact cohomology sequence of the exact triangle (3.1) it is therefore clear that $H^n_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ vanishes for all $n > 3$. But by Lemma 3.1.6 one also knows that the space $H^0_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ vanishes and so the complex $R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ is acyclic outside of degrees 1, 2, and 3.

Next we recall from Fukaya and Kato [30, Prop. 2.1.3] that the Euler characteristic of $R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ in $K_0(\mathbb{Q}_p[G])$ is zero. Since the algebra $\mathbb{Q}_p[G]$ is semisimple this implies that classes $[H^2_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r)]$ and $[H^1_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))\oplus H^3_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))]$ coincide in $K_0(\mathbb{Q}_p[G])$ and hence that $H^2_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ is isomorphic as a $\mathbb{Q}_p[G]$-module to the direct sum $H^1_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))\oplus H^3_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$. We note in particular that, for any character χ of G, the idempotent e_χ annihilates $H^2_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ if and only if it annihilates both $H^1_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ and $H^3_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$. In the sequel we fix such a character χ.

Now if $r\neq 0$, then the space $H^0(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ is clearly zero and so the long exact cohomology sequence of the distinguished triangle (3.1) induces an injection $P^0(\mathcal{O}_{K,S},\mathbb{Q}_p(r))\hookrightarrow H^1_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$. Thus e_χ also annihilates $P^0(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$. But by Lemma 4.2.2 below the (non-zero) $\mathbb{C}_p[G]$-module $P^0(\mathcal{O}_{K,S},\mathbb{Q}_p(r))\otimes_{\mathbb{Q}_p}\mathbb{C}_p$ is free over $\mathbb{C}_p[G]e_r^-$ and so $P^0(\mathcal{O}_{K,S},\mathbb{Q}_p(r))e_\chi$ is not trivial whenever χ satisfies $e_\chi e_r^- = e_\chi$. This therefore implies claim (i) whenever $r\neq 0$.

If now $r = 0$, then the Artin-Verdier duality theorem (cf. [43, Chap. II, Cor. 3.2]) combines with Kummer theory to induce a natural isomorphism of $\mathbb{Q}_p[G]$-modules between $\mathrm{Hom}_{\mathbb{Q}_p}(H^2_c(\mathcal{O}_{K,S}, \mathbb{Q}_p), \mathbb{Q}_p)$ and $H^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(1)) \cong \mathcal{O}_{K,S}^\times \otimes_{\mathbb{Z}} \mathbb{Q}_p$. But by [63, Thm. 4.12] the module $(\mathcal{O}_{K,S}^\times \otimes_{\mathbb{Z}} \mathbb{Q}_p)e_0^+$ is zero and so $(\mathcal{O}_{K,S}^\times \otimes_{\mathbb{Z}} \mathbb{Q}_p)e_0^- = (\mathcal{O}_{K,S}^\times \otimes_{\mathbb{Z}} \mathbb{Q}_p)$ which is clearly non-zero. In particular, if e_χ annihilates $H^2_c(\mathcal{O}_{K,S}, \mathbb{Q}_p)$, and hence also $\mathrm{Hom}_{\mathbb{Q}_p}(H^2_c(\mathcal{O}_{K,S}, \mathbb{Q}_p), \mathbb{Q}_p)$, then it must satisfy $e_\chi e_0^+ = e_\chi$, as required to complete the proof of claim (i).

In the following result we use the $G_k \times G$-module $\mathbb{Z}_p(r)_K := \mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G]$ introduced in §2.2.1. (We note that the explicit description in claim (ii) of this result will also be very useful in Chapter 5).

Lemma 4.2.2.

(i) If $r \neq 0$, then there is a natural isomorphism in $\mathcal{D}^p(\mathbb{Z}_p[G])$ of the form

$$\bigoplus_{w|\infty} R\Gamma(K_w, \mathbb{Z}_p(r)) \cong P^0(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))[0] \cong (\mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K)^+[0]$$

where the superscript $+$ denotes $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants.

(ii) Set $d := [k : \mathbb{Q}]$ and write $\tau_1, \ldots, \tau_d$ for the d distinct embeddings of k into $\overline{\mathbb{Q}}$. For each index i fix an element $\widetilde{\tau}_i$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which extends τ_i. Then for any integer r there is an isomorphism of $\mathbb{Z}_p[G]$-modules of the form

$$(\mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K)^+ \cong \bigoplus_{\widetilde{\tau}_i|_K} \mathbb{Z}_p[G]e_r^-,$$

where the action of G on the right-hand term is diagonal by precomposition with the indexing maps and multiplication on the coefficients.

Proof. For each place v of k Shapiro's lemma induces a canonical isomorphism in

$\mathcal{D}^p(\mathbb{Z}_p[G])$ of the form

$$\bigoplus_{w|v} R\Gamma(K_w, \mathbb{Z}_p(r)) \cong R\Gamma(k_v, \mathbb{Z}_p(r)_K)$$

where the direct sum is over all places of K lying over v. In particular, by considering archimedean places and taking cohomology in degree zero this gives a canonical isomorphism of $\mathbb{Z}_p[G]$-modules

$$\begin{aligned} P^0(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) = \bigoplus_{w|\infty} H^0(K_w, \mathbb{Z}_p(r)) &\cong \bigoplus_{v|\infty} H^0(k_v, \mathbb{Z}_p(r)_K) \\ &\cong \bigoplus_{v|\infty} (\mathbb{Z}_p(r)_K)^{G_v} \\ &\cong \left(\mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K\right)^+, \end{aligned}$$

where the first equality is because $H^0(K_w, \mathbb{Z}_p(r))$ vanishes for each non-archimedean place w (since $r \neq 0$). This proves the second isomorphism of claim (i). To prove the first isomorphism we recall that for such archimedean place v the decomposition group G_v is generated by a unique element c_v (of order 2). Thus, since p is assumed to be odd, it is a simple calculation in group cohomology for the group G_v and topological G_v-module $\mathbb{Z}_p(r)_K$ to show that $H^i(k_v, \mathbb{Z}_p(r)_K)$ vanishes for each $i > 0$, as required to complete the proof of claim (i).

To prove claim (ii) we first fix a topological generator $\zeta := (\exp(2\pi i/p^n)^{\otimes r})_{n\geq 0}$ of $\mathbb{Z}_p(r)$ and denote the elements of $\mathrm{Emb}(K, \overline{Q})$ by σ_s for $1 \leq s \leq [K : \mathbb{Q}]$. We then obtain a $\mathbb{Z}_p$-basis $\{\zeta_{\sigma_s} : 1 \leq s \leq [K : \mathbb{Q}]\}$ of $\bigoplus_{K\hookrightarrow\overline{\mathbb{Q}}} \mathbb{Z}_p(r)$ by setting

$$\zeta_{\sigma_s} := \begin{cases} \zeta & \text{at the embedding } \sigma_s \\ 0 & \text{elsewhere.} \end{cases}$$

We can now define an explicit homomorphism

$$\begin{aligned} \mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K = \bigoplus_{K \hookrightarrow \overline{\mathbb{Q}}} \mathbb{Z}_p(r) &\cong \bigoplus_{\widetilde{\tau}_i|_K} \mathbb{Z}_p[G] \\ \zeta_{\sigma_s} &\mapsto (g_t)_{\widetilde{\tau}_i} \end{aligned} \tag{4.7}$$

where (we recall G acts on the second term by precomposition with the indexing maps and) the embedding σ_s corresponds to the pair (τ_i, g_t) via the bijection

$$\begin{array}{rcl} \mathrm{Emb}(k, \overline{\mathbb{Q}}) \times G & \cong & \mathrm{Emb}(K, \overline{\mathbb{Q}}) \\ (\tau_i, g_t) & \mapsto & (\widetilde{\tau}_i|_K) \circ g_t. \end{array} \tag{4.8}$$

It is straightforward to check that (4.7) is an isomorphism of $\mathbb{Z}_p[G]$-modules. We now write τ for complex conjugation. Then, as p is odd, the submodule $(\mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K)^+$ is equal to the image of $\frac{1}{2}(1+\tau)$ on $\mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K$. Thus, since τ sends the element ζ_{σ_s} to $(-1)^r \zeta_{\tau\circ\sigma_s}$, $\tau \circ \sigma_s = \sigma_s \circ \tau_K$ and $e_r^- = \frac{1}{2}(1 + (-1)^r \tau_K)$, it is easy to check that (4.7) restricts to give an isomorphism of $\mathbb{Z}_p[G]$-modules of the form described in claim (ii). $\square$

4.2.3 The proof of Theorem 4.2.1(ii) and (iii)

We shall deduce Theorem 4.2.1(ii) & (iii) from Wiles's proof of the main conjecture of Iwasawa theory for totally real fields by copying arguments used by Burns and Greither in [14, 15]. We shall therefore first introduce (or recall) some convenient notation.

- Write E^∞ for the cyclotomic $\mathbb{Z}_p$-extension of a number field E. In particular, since K is a CM field so is K^∞ and there is a natural restriction isomorphism $\mathrm{Gal}(K^\infty/(K^\infty)^+) \cong \mathrm{Gal}(K/K^+)$. In the sequel we use this map to identify the groups $\mathrm{Gal}(K/K^+)$ and $\mathrm{Gal}(K^\infty/(K^\infty)^+)$ and hence regard the idempotent e_r^+ as belonging to both $\mathbb{Z}_p[G_{K/k}]$ and $\mathbb{Z}_p[G_{K^\infty/k}]$.

- For any topological quotient H of the abelian group $\Gamma := \mathrm{Gal}(K^\infty/k)$ let $\Lambda(H) := \varprojlim_U \mathbb{Z}_p\left[H/U\right]$ where U runs over all open (normal) subgroups of H.
- Write $Q(H)$ for the total quotient ring of $\Lambda(H)$ (this is a finite product of fields).
- Set $\Gamma_k := \mathrm{Gal}(k^\infty/k)$. Fix a topological generator γ of Γ_k and a pre-image $\widetilde{\gamma}$ of γ under the surjection $\Gamma \twoheadrightarrow \Gamma_k$.
- Write $K^{(s)}$ for the subfield of K^∞ which corresponds via Galois theory to the subgroup generated topologically by $\widetilde{\gamma}$. Set $\Gamma^{(s)} := \mathrm{Gal}(K^{(s)}/k)$.
- Identify Γ with the direct product $\Gamma^{(s)} \times \Gamma_k$ (which we may do, since both $K^{(s)} \cap k^\infty = k$ and $k^\infty K^{(s)} = K^\infty$). For each homomorphism $\psi : \Gamma \to \overline{\mathbb{Q}}_p^\times$ with open kernel there are thus unique homomorphisms $\psi^{(s)} : \Gamma^{(s)} \to \overline{\mathbb{Q}}_p^\times$ and $\psi^{(w)} : \Gamma_k \to \overline{\mathbb{Q}}_p^\times$ such that $\psi^{(w)}$ has open kernel and $\psi = \psi^{(s)} \times \psi^{(w)}$.
- Via the correspondence $\gamma \leftrightarrow T+1$ we identify $\Lambda(\Gamma)$ with the power series ring $\mathbb{Z}_p\left[\Gamma^{(s)}\right][[T]]$.
- For each integer n we write tw_n for the endomorphisms of $\Lambda(\Gamma)$ and $Q(\Gamma)$ that are induced by the map which sends each element g of Γ to $\chi_{\mathrm{cyclo}}(g)^n g \in \Lambda(\Gamma)$. We write $\Lambda(H)(n)$ for the Tate-twist of $\Lambda(H)$ by n – ie. the module $\Lambda(H) \otimes_{\Lambda(H),\mathrm{tw}_{H,n}} \Lambda(H)$ where $\Lambda(H)$ acts via multiplication on the left hand term and $\mathrm{tw}_{H,n}$ denotes the endomorphism of $\Lambda(H)$ induces by tw_n. (More explicitly, this means that $\Lambda(H)(n)$ can be identified with the set $\Lambda(H)$ upon which each element h of H acts as multiplicaiton by $\mathrm{tw}_{H,n}(h)$.)

In the next result we use the following fundamental construction of Deligne and Ribet. For any homomorphism $\psi : \Gamma \to \overline{\mathbb{Q}}_p^\times$ with open kernel there exists a power-series $G_{\psi,S}(T) \in \mathbb{Z}_p[\psi][[T]]$ such that

$$f_{S,\psi}(s) := \begin{cases} G_{\psi,S}(\chi_{\mathrm{cyclo}}(\gamma)^s - 1)/(\psi(\gamma)\chi_{\mathrm{cyclo}}(\gamma)^s - 1), & \text{if } \psi^{(s)} \text{ is trivial} \\ G_{\psi,S}(\chi_{\mathrm{cyclo}}(\gamma)^s - 1) & \text{otherwise,} \end{cases}$$

is equal to $L_{p,S}(\psi, 1-s)$ for almost all s (cf. [14, Eqn (1)]), in particular this equality holds when s is an integer other than 1 and when ψ is non-trivial. In fact, the methods of Deligne and Ribet also prove the existence of elements G_S and H of the Iwasawa algebra $\Lambda(\Gamma)$ such that for all such homomorphisms ψ with open kernel the element $\psi(G_S/H)$ is both well defined and equal to $f_{S,\psi}(0)$ (for more details in this regard see [14, §3]). We recall also that for each such homomorphism κ such that $\kappa^{(s)}$ is trivial one has $f_{S,\kappa\psi}(0) = f_{S,\psi}(\kappa(\gamma) - 1)$ (cf. [65, Eqn. (1.4)]).

Proposition 4.2.3. *Set $R := \Lambda(\Gamma)$, write $Q(R)$ for the total quotient ring of R and, for any integer r, set $C_r^\bullet := R\Gamma_c(\mathcal{O}_{k,S}, R(r)e_r^+)$.*

(i.) The complex $C_r^\bullet \otimes_R Q(R)$ is acyclic and hence there is a canonical composite morphism

$$[C_r^\bullet]^{-1}_{Re_r^+} \subseteq [C_r^\bullet]^{-1}_{Re_r^+} \otimes_R Q(R) \cong [C_r^\bullet \otimes_R Q(R)]^{-1}_{Q(R)e_r^+} \cong [0]^{-1}_{Q(R)e_r^+} = (Q(R)e_r^+, 0). \tag{4.9}$$

(ii.) If $r = 1$, then the element G_S/H generates the ungraded part of the image of $[C_1^\bullet]^{-1}_{Re_1^+}$ under (4.9).

(iii.) If K contains a primitive p^{th}-root of unity, then for any integer r the element $\mathrm{tw}_{1-r}(G_S/H)$ generates the ungraded part of the image of $[C_r^\bullet]^{-1}_{Re_r^+}$ under (4.9).

Proof. It is well known that $C_r^\bullet$ is acyclic outside of degrees 2 and 3 and that there are canonical isomorphisms $H^3(C_r^\bullet) \cong \mathbb{Z}_p(r-1)$ and $H^2(C_r^\bullet) \cong \mathcal{H}^+ \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r-1)$, where $\mathcal{H} = \mathrm{Gal}(M_S^p(K)/K^\infty)$ and Γ acts diagonally on the tensor product (cf. [14, Proof of Lem. 3]). Hence since both $\mathcal{H}^+$ and $\mathbb{Z}_p(1-r)$ are finitely generated torsion Re_r^+-modules it is clear that $C_r^\bullet \otimes_R Q(R)$ is acyclic. The existence of the composite morphism (4.9) in claim (i) is then also clear.

To prove claim (ii) we write Ξ_1 for the image of $[C_1^\bullet]_{Re_1^+}^{-1}$ under (4.9). Then the algebraic observation of Burns and Greither in [15, Lem. 6.1] implies that claim (ii) is valid provided that at every height one prime ideal $\mathfrak{p}$ of R the localisation $\Xi_{1,\mathfrak{p}} := \Xi_1 \otimes_R R_\mathfrak{p}$ is equal to $(G_S/H) \cdot R_\mathfrak{p} e_1^+$.

To prove this we first assume that $\mathfrak{p}$ is a height one prime ideal of R with residue characteristic p. Then the explicit description of the groups $H^a(C_1)$ given above combine with [15, Lem. 6.3] and the assumed vanishing of $\mu(K,p)$ to imply that the complex $C_1^\bullet \otimes_R R_\mathfrak{p}$ is acyclic and hence that $\Xi_{1,\mathfrak{p}} = R_\mathfrak{p} e_1^+$. The required equality is therefore valid in this case because the assumed vanishing of $\mu(K,p)$ also implies that both G_S and H are units in $R_\mathfrak{p} e_1^+$ and hence that $(G_S/H) \cdot R_\mathfrak{p} e_1^+ = R_\mathfrak{p} e_1^+$ (indeed, this follows directly from [14, Thm. 3.1 (iii)]).

We now assume that $\mathfrak{p}$ is a height one prime ideal of R with residue characteristic 0. Then $R_\mathfrak{p} e_1^+$ is both a discrete valuation ring and a $\mathbb{Q}_p$-algebra and decomposes as a product $\prod_\psi R_{\mathfrak{p},\psi}$ where ψ runs over all homomorphisms $\Gamma \to \overline{\mathbb{Q}}_p^\times$ such that $\psi(\tau) = 1$, where we write τ for the unique non-trivial element of $\mathrm{Gal}(K^\infty/(K^\infty)^+)$. Now in this decomposition each algebra $R_{\mathfrak{p},\psi}$ is a principle ideal domain and hence there exist canonical isomorphisms of graded $R_{\mathfrak{p}}e_1^+$-modules

$$\begin{aligned}[C_1^\bullet \otimes_R R_{\mathfrak{p}}]^{-1}_{R_{\mathfrak{p}}e_1^+} &\cong \left[\mathcal{H}^+ \otimes_R R_{\mathfrak{p}}\right]^{-1}_{R_{\mathfrak{p}}e_1^+} \otimes_{R_{\mathfrak{p}}e_1^+} [\mathbb{Z}_p \otimes_R R_{\mathfrak{p}}]_{R_{\mathfrak{p}}e_1^+} \\ &\cong (\mathrm{Fitt}_{R_{\mathfrak{p}}e_1^+}(\mathcal{H}^+ \otimes_R R_{\mathfrak{p}}), 0) \otimes_{R_{\mathfrak{p}}e_1^+} (\mathrm{Fitt}_{R_{\mathfrak{p}}e_1^+}(\mathbb{Z}_p \otimes_R R_{\mathfrak{p}})^{-1}, 0) \\ &= (\mathrm{Fitt}_{R_{\mathfrak{p}}e_1^+}(\mathcal{H}^+ \otimes_R R_{\mathfrak{p}}) \cdot \mathrm{Fitt}_{R_{\mathfrak{p}}e_1^+}(\mathbb{Z}_p \otimes_R R_{\mathfrak{p}})^{-1}, 0),\end{aligned}$$

where the first isomorphism comes from Lemma 2.1.4 and the fact that $C_1^\bullet \otimes_R R_{\mathfrak{p}}$ is cohomologically perfect and the second from Lemma 2.1.6(ii), and the equality is by definition of the product of graded invertible $R_{\mathfrak{p}}e_1^+$-modules. To complete the proof of claim (ii) it is therefore enough to prove that for each homomorphism $\psi : \Gamma \to \overline{\mathbb{Q}}_p^\times$ with $\psi(\tau) = 1$ the fractional $R_{\mathfrak{p},\psi}$-ideal $\mathrm{Fitt}_{R_{\mathfrak{p},\psi}}(e_\psi(\mathcal{H}^+ \otimes_R R_{\mathfrak{p}}))\,\mathrm{Fitt}_{R_{\mathfrak{p},\psi}}(e_\psi(\mathbb{Z}_p \otimes_R R_{\mathfrak{p}}))^{-1}$ is generated by $f_{S,\psi}(0)$. But if $\psi^{(s)}$ is trivial, then it is clear that $\mathrm{Fitt}_{R_{\mathfrak{p},\psi}}(e_\psi(\mathbb{Z}_p \otimes_R R_{\mathfrak{p}})) = R_{\mathfrak{p},\psi} \cdot \psi(I_{\Gamma_k}) = R_{\mathfrak{p},\psi} \cdot (\psi(\gamma) - 1)$. If $\psi^{(s)}$ is non-trivial, then $e_\psi(\mathbb{Z}_p \otimes_R R_{\mathfrak{p}})$ vanishes and so $\mathrm{Fitt}_{R_{\mathfrak{p},\psi}}(e_\psi(\mathbb{Z}_p \otimes_R R_{\mathfrak{p}})) = R_{\mathfrak{p},\psi}$. Further, the main conjecture of Iwasawa theory proved by Wiles in [65] states precisely that in this case $\mathrm{Fitt}_{e_\psi R_{\mathfrak{p}}}(e_\psi(\mathcal{H}^+ \otimes_R R_{\mathfrak{p}})) = R_{\mathfrak{p},\psi} \cdot G_{S,\psi^{(s)}}(\psi(\gamma) - 1)$ (cf. [14, (3)]), and so claim (ii) follows.

To prove claim (iii) we note that for each integer r there are isomorphisms of R-modules of the form

$$R(r)e_r^+ \cong R \otimes_{R,\mathrm{tw}_{r-1}} R(1)e_1^+ \cong \mathbb{Z}_p(r-1) \otimes_{\mathbb{Z}_p} R(1)e_1^+, \tag{4.10}$$

where $\Gamma \subset R$ acts on the first tensor product via multiplication on the left hand term and acts diagonally on the second tensor product. (Indeed, the first displayed isomorphism is a simple consequence of the fact that $\mathrm{tw}_r = \mathrm{tw}_{r-1} \circ \mathrm{tw}_1$ and the second is induced by sending $1 \otimes_R x$ to $\zeta^{\otimes(r-1)} \otimes_{\mathbb{Z}_p} x$.) Now, since K is assumed to contain a primitive p^{th}-root of unity, the same reasoning that produces the isomorphism of

[14, Eqn (10)] shows that the isomorphism (4.10) induces an isomorphism in $\mathcal{D}^p(R)$ of the form

$$R \otimes_{R,\mathrm{tw}_{r-1}} C_1^\bullet \cong C_r^\bullet.$$

These isomorphisms combine with the fact that determinants commute with scalar extension to induce an isomorphism of graded Re_r^+-modules of the form

$$[C_r^\bullet]^{-1}_{Re_r^+} \cong \left[R \otimes_{R,\mathrm{tw}_{r-1}} C_1^\bullet\right]^{-1}_{Re_r^+} \cong \left(R \otimes_{R,\mathrm{tw}_{r-1}} [C_1^\bullet]_{Re_1^+}\right)^{-1} \cong R \otimes_{R,\mathrm{tw}_{1-r}} ([C_1^\bullet]_{Re_1^+})^{-1}.$$

Upon comparing these isomorphisms to those involved in (4.9) one finds that the image of $[C_r^\bullet]^{-1}_{Re_r^+}$ under the composite map (4.9) is equal to $\mathrm{tw}_{1-r}(\Xi_1)$. Hence claim (ii) implies claim (iii) and so completes the proof of Proposition 4.2.3. $\square$

In the rest of this section we shall show that Theorem 4.2.1(ii) and (iii) are consequences of Proposition 4.2.3. To do this we set $R := \Lambda(\Gamma)$, $\mathfrak{A} := \mathbb{Z}_p[G]e_r^{(0)}$, $C_\infty^\bullet := R\Gamma_c(\mathcal{O}_{k,S}, R(r)e_r^+)$ and $C^\bullet := R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^+$. Then there is a canonical isomorphism in $\mathcal{D}^p(\mathbb{Z}_p[G]e_r^+)$ of the form $\mathbb{Z}_p[G] \otimes^{\mathbb{L}}_R C_\infty^\bullet \cong R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^+$ (induced by combining the 'descent isomorphism' of [15, §3.1] with Shapiro's lemma) which in turn induces an isomorphism of graded $\mathfrak{A}$-modules

$$[C_\infty^\bullet]^{-1}_{Re_r^+} \otimes_R \mathfrak{A} \cong \left[C^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}\right]^{-1}_{\mathfrak{A}}. \tag{4.11}$$

We also note that the complex $(C^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is acyclic. Indeed, this is a consequence of the definition of the idempotent $e_r^{(0)}$ in the statement of Theorem 4.2.1 and the fact that if an idempotent annihilates $H_c^2(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$, then it annihilates $H_c^i(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ in all degrees i (cf. the second paragraph of §4.2.2).

In particular, if $r = 1$, then the isomorphism (4.11) combines with Proposition 4.2.3(ii), the acylicity of $(C^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the descent formula of [15, Lem. 8.1] to imply that

$$\left[C^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}\right]^{-1}_{\mathfrak{A}} = (\mathbb{Z}_p[G]\mathfrak{L}_1, 0)$$

where $\mathfrak{L}_1$ is the (unique) element of $\mathbb{Q}_p[G]e_1^{(0)}$ with the following property: for each homomorphism $\phi : G \to \overline{\mathbb{Q}}_p^\times$ with $\phi(e_1^{(0)}) = 1$ one has $\phi(\mathfrak{L}_1) = \phi(G_S/H)$. In particular if ϕ is non-trivial, then one has $\phi(\mathfrak{L}_1) = f_{S,\phi}(0) = L_{p,S}(\phi, 1)$ and so claim (ii) of Theorem 4.2.1 holds.

Claim (iii) of Theorem 4.2.1 follows in a similar way. Indeed, if K contains a primitive p^{th} root of unity, then (4.11) combines with Proposition 4.2.3(iii), the acylicity of $(C^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the descent formula of [15, Lem. 8.1] to imply that

$$\left[C^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}\right]^{-1}_{\mathfrak{A}} = (\mathbb{Z}_p[G]\mathfrak{L}_r, 0)$$

where $\mathfrak{L}_r$ is the (unique) element of $\mathbb{Q}_p[G]e_r^{(0)}$ with the following property: for each homomorphism $\phi : G \to \overline{\mathbb{Q}}_p^\times$ with $\phi(e_r^{(0)}) = 1$ one has $\phi(\mathfrak{L}_r) = \phi(\mathrm{tw}_{1-r}(G_S/H))$. Hence for each such ϕ one has

$$\begin{aligned}\phi(\mathfrak{L}_r) = (\chi_{cyc}^{1-r}\phi)(G_S/H) = f_{S,\chi_{cyc}^{1-r}\phi}(0) \\ = f_{S,\omega^{1-r}\phi}(\chi_{cyc}(\gamma)^{1-r} - 1) \\ = L_{p,S}(\omega^{1-r}\phi, r) = \phi(\mathfrak{L}_{p,S}(r)),\end{aligned}$$

where the first equality follows from the definition of tw_{1-r}, the second from the definition of G_S and H, the third from the fact that the Teichmüller character ω is such that $\omega = \omega^{(s)} = (\chi_{cyc})^{(s)}$, the fourth is a fundamental interpolation property of the function $f_{S,\omega^{1-r}\phi}(s)$ (cf. [65, Eqns. (1.4) and (1.3)]), and the last follows from the definition of $\mathfrak{L}_{p,S}(r)$. This completes the proof of claim (iii) of Theorem 4.2.1 and hence of Theorem 4.2.1 itself.

4.3 The Key Algebraic Result

This section is joint work with David Burns.

In this section we prove an algebraic annihilation result which allows us to deduce Theorems 4.1.1 and 4.1.5 from Theorem 4.2.1, and which is also surely itself of some independent interest.

Throughout this section we fix a finite abelian group G and a direct factor $\mathfrak{A}$ of $\mathbb{Z}_p[G]$. We will often (and usually without explicit comment) use the fact that the associated $\mathbb{Q}_p$-algebra $A := \mathfrak{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semisimple.

4.3.1 Statement of the Result

We shall first introduce the general class of complexes to which our annihilation result applies. (In §4.4 we will see that such complexes in fact arise naturally in the context of Theorem 4.2.1.)

Definition 4.3.1.

- We call a (co-chain) complex of $\mathfrak{A}$-modules $C^\bullet$ *weakly admissible* if it satisfies the following conditions:

  (i) $C^\bullet$ is a perfect complex of $\mathfrak{A}$-modules.

  (ii) The Euler characteristic of $C^\bullet \otimes_{\mathfrak{A}} A$ in $K_0(A)$ is zero.

  (iii) $C^\bullet$ is acyclic outside of degrees 1, 2 and 3.

(iv) $H^1(C)_{\mathrm{tor}}$ is a perfect $\mathfrak{A}$-module.

- We call a weakly admissible complex $C^\bullet$ *admissible* if $H^1(C)_{\mathrm{tor}} = 0$.

- We call an admissible complex $C^\bullet$ *strongly admissible* if $H^1(C) = 0$.

For any perfect complex of $\mathfrak{A}$-modules $E^\bullet$ we also introduce the following notation:

- Set $e^{(0)} = e^{(0)}_{E^\bullet}$ to be the sum of all primitive idempotents of A which annihilate the module $H^2(C^\bullet \otimes_{\mathfrak{A}} A)$.

- Set $\mathfrak{A}^{(0)} = \mathfrak{A}^{(0)}_{E^\bullet} := \mathfrak{A}e^{(0)} \subseteq A^{(0)} := Ae^{(0)}$.

- Set $I^{(0)} = I^{(0)}_{E^\bullet} := \mathfrak{A} \cap \mathfrak{A}^{(0)}$.

- Set $E^\bullet_{(0)} := E^\bullet \otimes^{\mathbb{L}}_{\mathfrak{A}} \mathfrak{A}^{(0)}$.

Remark 4.3.2. If C is weakly admissible then, since A is semisimple, the above conditions (ii) and (iii) combine to imply that the A-modules $H^2(C^\bullet \otimes_{\mathfrak{A}} A)$ and $H^1(C^\bullet \otimes_{\mathfrak{A}} A) \oplus H^3(C^\bullet \otimes_{\mathfrak{A}} A)$ are isomorphic. In this case the idempotent $e^{(0)}$ can therefore also be defined as the sum of all primitive idempotents of A which annihilate the module $H^i(C^\bullet \otimes_{\mathfrak{A}} A) = H^i(C^\bullet) \otimes_{\mathfrak{A}} A$ for all degrees i.

If $C^\bullet$ is weakly admissible, then the complex $C^\bullet_{(0)}$ is a perfect complex of $\mathfrak{A}^{(0)}$-modules and in the following result we identify the determinant module $\left[C^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}}$ with a graded fractional $\mathfrak{A}^{(0)}$-ideal via the composite inclusion

$$\left[C^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}} \subseteq \left[C^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}} \otimes_{\mathfrak{A}^{(0)}} A^{(0)} \cong \left[C^\bullet_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}\right]_{A^{(0)}} \cong [0]_{A^{(0)}} = (A^{(0)}, 0). \quad (4.12)$$

We recall that a $\mathbb{Z}_p$-order $\mathfrak{B}$ is said to be 'relatively Gorenstein over $\mathbb{Z}_p$' (or more simply 'self-dual') if $\mathfrak{B}^*$ is isomorphic to $\mathfrak{B}$ as a $\mathfrak{B}$-module. We can now state the main result of this section.

Theorem 4.3.3. *For any complex of $\mathfrak{A}$-modules $D^\bullet$ we write $g(D^\bullet)$ for the minimal number of generators of the $\mathfrak{A}$-module $H^3(D^\bullet)$ and define $n(D^\bullet) = 1$ if $H^1(D^\bullet)_{\mathrm{tf}} = 0$ and $n(D^\bullet) = 2$ otherwise.*

Let $C^\bullet$ be a weakly admissible complex of $\mathfrak{A}$-modules.

(i) One has

$$\left((I^{(0)})^{g(C^\bullet)+n(C^\bullet)}\,\mathrm{Fitt}_{\mathfrak{A}}(H^1(C^\bullet)_{\mathrm{tor}})\,\mathrm{Ann}_{\mathfrak{A}}(H^3(C^\bullet)_{\mathrm{tor}})^{g(C^\bullet)}, 0\right) \otimes_{\mathfrak{A}} \left[C^\bullet_{(0)}\right]^{-1}_{\mathfrak{A}^{(0)}} \subseteq \left(\mathrm{Ann}_{\mathfrak{A}}(H^2(C^\bullet)_{\mathrm{tor}}), 0\right).$$

(ii) One has

$$\left((I^{(0)})^{g(C^*)+n(C^*)}\,\mathrm{Ann}_{\mathfrak{A}}(H^2(C^\bullet)_{\mathrm{tor}})^{g(C^*)}, 0\right) \otimes_{\mathfrak{A}} \left[C^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}} \subseteq \left(\mathrm{Fitt}_{\mathfrak{A}}(H^1(C^\bullet)_{\mathrm{tor}})\,\mathrm{Ann}_{\mathfrak{A}}(H^3(C^\bullet)_{\mathrm{tor}}), 0\right)$$

where $C^* := R\,\mathrm{Hom}_{\mathbb{Z}_p}(C^\bullet, \mathbb{Z}_p[-4])$.

(iii) If $C^\bullet$ is strongly admissible and in addition either $H^2(C^\bullet)_{\mathrm{tor}} = 0$ or $\mathfrak{A}^{(0)}$ is relatively Gorenstein over $\mathbb{Z}_p$, then

$$\left(\mathrm{Fitt}_{\mathfrak{A}}((H^2(C^\bullet)_{\mathrm{tor}})^\vee), 0\right) \otimes_{\mathfrak{A}} \left[C^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}} = \left(\mathrm{Fitt}_{\mathfrak{A}}(H^3(C^\bullet)), 0\right).$$

Remark 4.3.4. Theorem 4.3.3 generalises Snaith's result [51, Thm. 7.1.11]. Indeed to recover that result one need only apply claims (i) and (ii) of Theorem 4.3.3 with $\mathfrak{A} = \mathbb{Z}_p[G]$ for a strongly admissible complex $C^\bullet$ (then adjust for differences in notation). If, in addition, $H^3(C^\bullet)$ is a cyclic $\mathbb{Z}_p[G]$-module then claim (iii) is strictly stronger than claims (i) and (ii) and hence also stronger than the Therorem of *loc. cit.*.

4.3.2 Dualising

Before embarking on the proof of Theorem 4.3.3 we first establish some useful facts concerning the duals of perfect complexes and modules.

In the sequel we shall use the following convenient notation: if $D^\bullet$ is any perfect complex of $\mathfrak{A}^{(0)}$-modules whose cohomology groups are finite (or, equivalently, $\mathbb{Z}_p$-torsion) in every degree, then we write $\Xi(D^\bullet)$ for the ungraded part of the image of $[D^\bullet]_{\mathfrak{A}^{(0)}}$ under the composite morphism $[D^\bullet]_{\mathfrak{A}^{(0)}} \to (A^{(0)}, 0)$ that is analogous to (4.12).

Lemma 4.3.5. *Let $\mathfrak{A}$ and A be as above. Let $E^\bullet$ be a perfect complex of $\mathfrak{A}$-modules and set $E^* := R\operatorname{Hom}_{\mathbb{Z}_p}(E^\bullet, \mathbb{Z}_p[-4])$.*

(i) In each degree i there are canonical isomorphisms $H^i(E^*)_{\mathrm{tf}} \cong H^{4-i}(E^\bullet)^*$ and $H^i(E^*)_{\mathrm{tor}} \cong (H^{5-i}(E^\bullet)_{\mathrm{tor}})^\vee$.

(ii) If $E^\bullet$ is weakly admissible, then E^* has the following properties: it is a perfect complex of $\mathfrak{A}$-modules; the Euler characteristic of $E^* \otimes_{\mathfrak{A}} A$ in $K_0(A)$ is zero; it is acyclic outside degrees $1, 2, 3$ and 4; $H^1(E^*)$ is torsion-free and $H^4(E^*) \cong (H^1(E^\bullet)_{\mathrm{tor}})^\vee$ is a finite perfect $\mathfrak{A}$-module with $\mathrm{Fit}_{\mathfrak{A}}(H^4(E^*)) = \mathrm{Fit}_{\mathfrak{A}}(H^1(E^\bullet)_{\mathrm{tor}})$.

(iii) If $E^\bullet$ is admissible, then E^* is admissible.

(iv) In each degree i an idempotent of A annihilates the module $H^i(E^* \otimes_{\mathfrak{A}} A)$ if and only if it annihilates the module $H^{4-i}(E^\bullet \otimes_{\mathfrak{A}} A)$.

(v) If $E^\bullet_{(0)}$ has finite cohomology groups, then $E^*_{(0)}$ has finite cohomology groups and $\Xi(E^*_{(0)}) = \Xi(E^\bullet_{(0)})^{-1}$.

Proof. Claim (i) is a consequence of the fact that for each perfect complex of $\mathfrak{A}$-modules $E^\bullet$ there is a cohomological spectral sequence of the form

$$\mathrm{Ext}^a_{\mathbb{Z}_p}(H^b(E^\bullet), \mathbb{Z}_p) \implies H^{a-b}(R\,\mathrm{Hom}_{\mathbb{Z}_p}(E^\bullet, \mathbb{Z}_p)) = H^{a-b+4}(E^*).$$

Indeed, since for every finitely generated $\mathbb{Z}_p$-module M the group $\mathrm{Ext}^a_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ vanishes for all $a \notin \{0, 1\}$ the second sheet of this spectral sequence has only two non-zero rows and so for any integer n there is a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}_p}(H^{5-n}(E^\bullet), \mathbb{Z}_p) \to H^n(E^*) \to \mathrm{Hom}_{\mathbb{Z}_p}(H^{4-n}(E^\bullet), \mathbb{Z}_p) \to 0 \tag{4.13}$$

(cf. [64, Excercise 5.2.2]). To deduce claim (i) from here we need only note that for any finitely generated $\mathbb{Z}_p$-module M there are natural identifications $\mathrm{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \cong \mathrm{Ext}^1_{\mathbb{Z}_p}(M_{\mathrm{tor}}, \mathbb{Z}_p) \cong (M_{\mathrm{tor}})^\vee$.

Since $\mathfrak{A}$ is a direct factor of $\mathbb{Z}_p[G]$ it is isomorphic to $\mathfrak{A}^*$. This implies that a finitely generated $\mathfrak{A}$-module M is projective, resp. free, if and only if its linear dual M^* is projective, resp. free. The first two assertions in claim (ii) concerning E^* are therefore clear. Further, since $E^\bullet$ is acyclic outside degrees $1, 2$ and 3, claim (i) implies that E^* is acyclic outside degrees $1, 2, 3$ and 4, that $H^1(E^*)_{\mathrm{tor}} \cong (H^4(E^\bullet)_{\mathrm{tor}})^\vee$ and $H^4(E^*)_{\mathrm{tf}} \cong H^0(E^\bullet)^*$ both vanish and that $H^4(E^*) = H^4(E^*)_{\mathrm{tor}}$ is isomorphic to $(H^1(E^\bullet)_{\mathrm{tor}})^\vee$. To complete the proof of claim (ii) it is therefore enough to prove that if M is any finite perfect $\mathfrak{A}$-module (such as $H^1(E^\bullet)_{\mathrm{tor}}$) then $M^\vee$ is a finite perfect $\mathfrak{A}$-module and $\mathrm{Fit}_{\mathfrak{A}}(M^\vee) = \mathrm{Fit}_{\mathfrak{A}}(M)$ and the latter fact is proved by Burns and Greither in [14, Lem. 7].

Claim (iii) follows directly from claim (ii), the definition of admissibility and the fact that (since $E^\bullet$ is admissible) the group $H^4(E^*) \cong (H^1(E^\bullet)_{\mathrm{tor}})^\vee$ vanishes.

In each degree i claim (ii) induces an isomorphism of A-modules of the form $H^i(E^* \otimes_{\mathfrak{A}} A) \cong \mathrm{Hom}_{\mathbb{Q}_p}(H^{4-i}(E^\bullet \otimes_{\mathfrak{A}} A), \mathbb{Q}_p)$ and this implies claim (iv).

Regarding claim (v) we note first that the complex $E^\bullet_{(0)}$ has finite cohomology groups precisely when the idempotent $e^{(0)}$ annihilates $H^i(E^\bullet \otimes_{\mathfrak{A}} A)$ in all degrees i. In particular, from claim (iv) we deduce that if $E^\bullet_{(0)}$ has finite cohomology groups, then so does $E^*_{(0)}$. The claimed equality $\Xi(E^*_{(0)}) = \Xi(E^\bullet_{(0)})^{-1}$ is then also a consequence of fact that if P is any finitely generated projective $\mathfrak{A}$-module, then the natural isomorphism $P^* \cong \mathrm{Hom}_{\mathbb{Z}_p[G]}(P, \mathbb{Z}_p[G]) = \mathrm{Hom}_{\mathfrak{A}}(P, \mathfrak{A})$ induces an isomorphism of $\mathfrak{A}^{(0)}$-modules of the form $P^* \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \cong \mathrm{Hom}_{\mathfrak{A}}(P, \mathfrak{A}) \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \cong \mathrm{Hom}_{\mathfrak{A}^{(0)}}(P \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}, \mathfrak{A}^{(0)})$. Indeed, the required equality follows from this because for any graded invertible $\mathfrak{A}^{(0)}$-module (I, m) the ungraded part of $(I, m)^{-1}$ is (by definition) equal to $\mathrm{Hom}_{\mathfrak{A}^{(0)}}(I, \mathfrak{A}^{(0)})$. □

4.3.3 A useful reduction step

In this section we begin the proof of Theorem 4.3.3 by reducing to consideration of admissible complexes.

Lemma 4.3.6. *Theorem 4.3.3*(i)*, respectively* (ii)*, is valid provided that for every admissible complex $C^\bullet$ of $\mathfrak{A}$-modules there is an inclusion*

$$(I^{(0)})^{g(C^\bullet)+n(C^\bullet)} \, \mathrm{Ann}_{\mathfrak{A}}(H^3(C^\bullet)_{\mathrm{tor}})^{g(C^\bullet)} \cdot \Xi(C^\bullet_{(0)})^{-1} \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2(C^\bullet)_{\mathrm{tor}}), \tag{4.14}$$

respectively

$$(I^{(0)})^{g(C^*)+n(C^*)} \, \mathrm{Ann}_{\mathfrak{A}}(H^2(C^\bullet)_{\mathrm{tor}})^{g(C^*)} \cdot \Xi(C^\bullet_{(0)}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^3(C^\bullet)_{\mathrm{tor}}). \tag{4.15}$$

Proof. Let $D^\bullet$ be a weakly admissible complex of $\mathfrak{A}$-modules. Then since $D^\bullet$ is acyclic in degrees less than one the inclusion $H^1(D^\bullet)_{\mathrm{tor}} \subseteq H^1(D^\bullet)$ induces a natural morphism $(H^1(D^\bullet)_{\mathrm{tor}})[-1] \to D^\bullet$ in $\mathcal{D}^p(\mathfrak{A})$. Let $C^\bullet$ denote the mapping cone of this morphism, which by definition lies in a distinguished triangle

$$H^1(D^\bullet)_{\mathrm{tor}}[-1] \to D^\bullet \to C^\bullet \to . \tag{4.16}$$

Now the complexes $H^1(D^\bullet)_{\mathrm{tor}}[-1]$ and $D^\bullet$ are, by assumption, both objects in $\mathcal{D}^p(\mathfrak{A})$ and so this triangle implies that $C^\bullet$ also belongs to $\mathcal{D}^p(\mathfrak{A})$. Further, the long exact sequence of cohomology of (4.16) induces isomorphisms $H^n(D^\bullet) \cong H^n(C^\bullet)$ for all integers $n \neq 1$ and a short exact sequence

$$0 \to H^1(D^\bullet)_{\mathrm{tor}} \to H^1(D^\bullet) \to H^1(C^\bullet) \to 0$$

which in turn induces a canonical isomorphism $H^1(C^\bullet) \cong H^1(D^\bullet)_{\mathrm{tf}}$. The complex $C^\bullet$ is therefore admissible (with condition (ii) being satisfied because $C^\bullet \otimes_{\mathfrak{A}} A \cong D^\bullet \otimes_{\mathfrak{A}} A$).

Now the distinguished triangle (4.16) also combines with Lemma 2.1.6 to give an equality of graded $\mathfrak{A}$-modules $[D^\bullet]_{\mathfrak{A}} = [C^\bullet]_{\mathfrak{A}} \otimes_{\mathfrak{A}} (\mathrm{Fitt}_{\mathfrak{A}}(H^1(D^\bullet)_{\mathrm{tor}}), 0)$. Further, since the $\mathfrak{A}$-module $H^1(D^\bullet)_{\mathrm{tor}}$ has projective dimension at most one, the homomorphism $\mathrm{Fitt}_{\mathfrak{A}}(H^1(D^\bullet)_{\mathrm{tor}}) \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \to \mathrm{Fitt}_{\mathfrak{A}}(H^1(D^\bullet)_{\mathrm{tor}}) e^{(0)}$ which sends $x \otimes_{\mathfrak{A}} y$ to xy is bijective. Hence by applying $\otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}$ to the last equality we obtain an equality of graded $\mathfrak{A}^{(0)}$-modules $\left[D^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}} = \left[C^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}} \otimes_{\mathfrak{A}^{(0)}} (\mathrm{Fitt}_{\mathfrak{A}}(H^1(D^\bullet)_{\mathrm{tor}}) e^{(0)}, 0)$ and hence an equality of lattices

$$\Xi(C^\bullet_{(0)})^{-1} = \Xi(D^\bullet_{(0)})^{-1} \cdot \mathrm{Fitt}_{\mathfrak{A}}(H^1(D^\bullet)_{\mathrm{tor}}). \tag{4.17}$$

From the isomorphisms $H^1(C^\bullet)_{\mathrm{tf}} = H^1(C^\bullet) \cong H^1(D^\bullet)_{\mathrm{tf}}$ and $H^3(D^\bullet) \cong H^3(C^\bullet)$ we also have $g(C^\bullet) = g(D^\bullet)$ and $n(C^\bullet) = n(D^\bullet)$. Upon substituting (4.17) into (4.14) one therefore obtains the statement of Theorem 4.3.3(i) with $C^\bullet$ replaced by $D^\bullet$.

To make the analogous reduction for Theorem 4.3.3(ii) we use the fact that there is a distinguished triangle in $\mathcal{D}^p(\mathfrak{A})$ of the form

$$E^*[-1] \to C^* \to D^* \to \tag{4.18}$$

with $E^\bullet := H^1(D^\bullet)_{\mathrm{tor}}[-1]$. Indeed, to obtain this triangle one need only apply the functor $R\,\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p[-4])$ to the triangle (4.16). Now Lemma 4.3.5(ii) implies that E^* is acyclic outside degree 4 and so the exact sequence of cohomology of (4.18) implies that $H^1(C^*) = H^1(D^*)$ so $n(C^*) = n(D^*)$, and $H^3(C^*)_{\mathrm{tf}} = H^3(D^*)_{\mathrm{tf}}$ so $g(C^*) = g(D^*)$. In addition, the same argument as used above with (4.16) applies to (4.18) to give an equality of lattices

$$\begin{aligned}\Xi(D^\bullet_{(0)}) = \Xi(D^*_{(0)})^{-1} = \Xi(C^*_{(0)})^{-1} \cdot \mathrm{Fitt}_{\mathfrak{A}}(H^4(E^*))&\\ = \Xi(C^*_{(0)})^{-1} \cdot \mathrm{Fitt}_{\mathfrak{A}}(H^1(D^\bullet)_{\mathrm{tor}}) = \Xi(C^\bullet_{(0)}) \cdot \mathrm{Fitt}_{\mathfrak{A}}(H^1(D^\bullet)_{\mathrm{tor}})&\end{aligned}$$

where the first and last equality follow from Lemma 4.3.5(v) and the third from the final assertion of Lemma 4.3.5(ii). Upon substituting the last displayed equality into (4.15) one therefore obtains the statement of Theorem 4.3.3(ii) with $C^\bullet$ replaced by $D^\bullet$. This completes the proof of Lemma 4.3.6. $\square$

4.3.4 The Proof of Theorem 4.3.3(i)

In this subsection we shall prove the inclusion (4.14) and hence complete the proof of Theorem 4.3.3(i). To do this we fix an admissible complex of $\mathfrak{A}$-modules $C^\bullet$ and for any integer n set $H^n := H^n(C^\bullet)$ and $H^n_{(0)} := H^n(C^\bullet_{(0)})$.

As a first step we prove the following result.

Proposition 4.3.7. $\mathrm{Ann}_{\mathfrak{A}^{(0)}}(H^3_{(0)})^{g(C^\bullet)} \cdot \Xi(C^\bullet_{(0)})^{-1} \subseteq \mathrm{Ann}_{\mathfrak{A}^{(0)}}(H^2_{(0)})$.

The proof of this proposition requires two subsidiary lemmas.

Lemma 4.3.8. *The complex $C^\bullet$ is isomorphic in $\mathcal{D}^p(\mathfrak{A})$ to a complex of finitely generated free $\mathfrak{A}$-modules of the form*

$$C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3$$

where C^1 occurs in degree 1 and the $\mathfrak{A}$-rank of C^3 is equal to $g(C^\bullet)$.

Proof. Since $C^\bullet$ is both perfect and acyclic outside of degrees 1, 2 and 3 a standard argument shows that it is quasi-isomorphic to a complex of the form $C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3$, where C^2 and C^3 are finitely generated free $\mathfrak{A}$-modules and C^1 is a finitely generated $\mathfrak{A}$-module with finite projective dimension. Furthermore by the argument of [20, Lem. 4.7] we may also construct this complex such that the $\mathfrak{A}$-rank $\mathrm{rk}_{\mathfrak{A}}(C^3)$ of C^3 is equal to the minimal number of generators of the $\mathfrak{A}$-module H^3.

Now because $C^\bullet$ is admissible one knows that $\mathrm{Ker}(d^1) = H^1$ is $\mathbb{Z}_p$-free. Since C^2 is also $\mathbb{Z}_p$-free we see that C^1 must be $\mathbb{Z}_p$-free. But any finitely generated $\mathfrak{A}$-module that is both $\mathbb{Z}_p$-free and of finite projective dimension is projective as an $\mathfrak{A}$-module (to see this one need only consider the module as a $\mathbb{Z}_p[G]$-module and then combine [1, Thms. 7, 8 & 9] to show that it is projective as a $\mathbb{Z}_p[G]$-module and hence as an $\mathfrak{A}$-module) and so it follows that C^1 is a finitely generated projective $\mathfrak{A}$-module. In addition since both $C^2 \otimes_{\mathfrak{A}} A$ and $C^3 \otimes_{\mathfrak{A}} A$ are free A-modules, and the Euler characteristic of $C^\bullet \otimes_{\mathfrak{A}} A$ in $K_0(A)$ is zero (since $C^\bullet$ is admissible) the A-module $C^1 \otimes_{\mathfrak{A}} A$ is itself also free. As $\mathfrak{A}$ is a product of local rings and C^1 is a projective $\mathfrak{A}$-module this then implies that C^1 is a free $\mathfrak{A}$-module. □

Lemma 4.3.9. *The complex $C^\bullet_{(0)}$ is acyclic outside of degrees 2 and 3 and the modules $H^2_{(0)}$ and $H^3_{(0)}$ are both finite. Furthermore there is a natural isomorphism $H^3_{(0)} \cong H^3 \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}$ and exact sequence*

$$H^1(G, H^1 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}) \to H^2_{(0)} \to (H^2 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G \to H^2(G, H^1 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}). \tag{4.19}$$

Proof. At the outset we note that in each degree a there is a natural isomorphism of the form $H^a_{(0)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong (H^a \otimes_{\mathfrak{A}} A) \otimes_A A^{(0)} \cong e^{(0)}(H^a \otimes_{\mathfrak{A}} A)$. From Remark 4.3.2 it therefore follows that each module $H^a_{(0)}$ is finite.

Next we use Lemma 4.3.8 to represent $C^\bullet_{(0)}$ by a complex of free $\mathfrak{A}^{(0)}$-modules of the form

$$C^1 \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \xrightarrow{d^1_{(0)}} C^2 \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \xrightarrow{d^2_{(0)}} C^3 \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \tag{4.20}$$

where we set $d^a_{(0)} := d^a \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}$ in each degree a. This representative makes it clear that $H^a_{(0)}$ vanishes at all degrees $a \notin \{1, 2, 3\}$ and that $H^3_{(0)} = \mathrm{cok}(d^2_{(0)})$ is canonically isomorphic to $\mathrm{cok}(d^2) \otimes_{\mathfrak{A}} A^{(0)} = H^3 \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}$. We now use the fact that for any projective $\mathfrak{A}$-module Q there is a natural isomorphism of the form

$$Q \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \cong H_0(G, Q \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}) \cong H^0(G, Q \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}) = (Q \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G.$$

Since the functor $M \mapsto (M \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G$ is left exact these isomorphisms induce an identification of $H^1_{(0)} = \ker(d^1_{(0)})$ with $(\ker(d^1) \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G = (H^1 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G \subseteq H^1 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}$. In particular, since both H^1 and $\mathfrak{A}^{(0)}$ are by assumption $\mathbb{Z}_p$-free, the finite module $H^1_{(0)}$ is $\mathbb{Z}_p$-free and therefore vanishes.

Finally we obtain the exact sequence (4.19) as the exact sequence of low degree terms of a convergent cohomological spectral sequence of the form

$$H^a(G, H^b \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}) \implies H^{a+b}_{(0)}.$$

Indeed, the existence of such a spectral sequence follows from the isomorphisms $H^0(G, Q \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}) \cong Q \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}$ mentioned above (cf. [64, §5]). $\square$

We can now return to the proof of Proposition 4.3.7. Following Lemmas 4.3.8 and 4.3.9 we are able to prove this result by using a simple adaptation of the argument given by Snaith in [51, §7.1]

Throughout we represent $C^{\bullet}_{(0)}$ by the complex (4.20). From Lemma 4.3.9 we recall that $C^{\bullet}_{(0)}$ has finite cohomology groups and is acyclic outside of degrees 2 and 3 and we note that this implies in particular that the differential $d^1_{(0)}$ is injective.

Now $g(C^{\bullet})$ is equal to the $\mathfrak{A}^{(0)}$-rank $r := \mathrm{rk}_{\mathfrak{A}^{(0)}}(C^3_{(0)})$ of the free $\mathfrak{A}^{(0)}$-module $C^3_{(0)} = C^3 \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}$, and so to prove Proposition 4.3.7 it is enough to show that

$$\mathrm{Ann}_{\mathfrak{A}^{(0)}}(H^3_{(0)})^r \cdot \Xi(C^{\bullet}_{(0)})^{-1} \subseteq \mathrm{Ann}_{\mathfrak{A}^{(0)}}(H^2_{(0)}). \tag{4.21}$$

Given any x in $\mathrm{Ann}_{\mathfrak{A}^{(0)}}(H^3_{(0)})$ we pick a pre-image $\widehat{x}$ of x under the natural surjection $\mathbb{Z}_p[G] \twoheadrightarrow \mathfrak{A}^{(0)}$ and then choose a large enough multiple n of $|H^2_{(0)}||H^3_{(0)}|$ to ensure that $n + \widehat{x}$ is a unit in $\mathbb{Q}_p[G]$. The projection $t = ne^{(0)} + x$ of $n + \widehat{x}$ is then a *unit* of $A^{(0)}$ which annihilates $H^3_{(0)}$. Since $ne^{(0)}$ annihilates $H^2_{(0)}$ the binomial theorem implies that (4.21) is true if we can prove that

$$t^r \cdot \Xi(C^{\bullet}_{(0)})^{-1} \subseteq \mathrm{Ann}_{\mathfrak{A}^{(0)}}(H^2_{(0)}). \tag{4.22}$$

To do this we fix a basis $\{z_i : 1 \leq i \leq r\}$ of $C^3_{(0)}$. Then each element tz_i is in the kernel of the natural projection $C^3_{(0)} \to \mathrm{Cok}(d^2_{(0)}) = H^3_{(0)}$ and so we may choose an element b_i of $C^2_{(0)}$ such that $d^2_{(0)}(b_i) = tz_i$. The associated homomorphism $\eta : C^3_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)} \to C^2_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}$ which sends each element z_i to $t^{-1}b_i$ is then a section to $d^2_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}$ which satisfies $\eta(tC^3_{(0)}) \subseteq C^2_{(0)} \subseteq C^2_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}$. Combining with $d^1_{(0)}$ we thereby obtain an $A^{(0)}$-module isomorphism

$$X : (C^3_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}) \oplus (C^1_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}) \to C^2_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}$$

which sends each (x, y) to $\eta(x) + d^1_{(0)}(y)$ for $x \in C^3_{(0)}$ and $y \in C^1_{(0)}$.

This in turn gives rise to a composite isomorphism

$$\begin{aligned}
\left[C^\bullet_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}\right]^{-1}_{A^{(0)}} =& \left[C^1_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}\right]_{A^{(0)}} \otimes_{A^{(0)}} \left[C^2_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}\right]^{-1}_{A^{(0)}} \\
& \otimes_{A^{(0)}} \left[C^3_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}\right]_{A^{(0)}} \\
\cong& \left[(C^3_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}) \oplus (C^1_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)})\right]_{A^{(0)}} \\
& \otimes_{A^{(0)}} \left[C^2_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}\right]^{-1}_{A^{(0)}} \\
\cong& [\mathrm{Im}(X)]_{A^{(0)}} \otimes_{A^{(0)}} \left[C^2_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}\right]^{-1}_{A^{(0)}} \\
\cong& [\mathrm{Cok}(X)]^{-1}_{A^{(0)}} \\
=& (A^{(0)}, 0),
\end{aligned} \tag{4.23}$$

where the first isomorphism is standard, the second is induced by X, the third is induced by the tautological exact sequence

$$0 \to \mathrm{Im}(X) \to C^2_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)} \to \mathrm{Cok}(X) \to 0$$

and the final equality is because $\mathrm{Cok}(X) = 0$. This composite isomorphism coincides of course with the canonical isomorphism $\left[C^\bullet_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}\right]^{-1}_{A^{(0)}} \cong (A^{(0)}, 0)$ that is induced by the acyclicity of the complex $C^\bullet_{(0)} \otimes_{\mathfrak{A}^{(0)}} A^{(0)}$ but the point is that the above description will allow us to keep track *integrally* of the maps involved.

To exploit this we let $T : C^3_{(0)} \to C^3_{(0)}$ denote the $\mathfrak{A}^{(0)}$-module homomorphism given by multiplication by t and observe that

$$\begin{aligned} t^r \left[C^3_{(0)}\right]_{\mathfrak{A}^{(0)}} &= \left(t^r \bigwedge\nolimits^r_{\mathfrak{A}^{(0)}} C^3_{(0)}, r\right) \\ &= \left(\bigwedge\nolimits^r_{\mathfrak{A}^{(0)}} (t \cdot C^3_{(0)}), r\right) \\ &= \left(\bigwedge\nolimits^r_{\mathfrak{A}^{(0)}} \mathrm{Im}(T), r\right) \\ &= [\mathrm{Im}(T)]_{\mathfrak{A}^{(0)}}. \end{aligned}$$

Now $\eta(tC^3_{(0)}) \subseteq C^2_{(0)}$ and so the composite isomorphism $X \circ ((T \otimes \mathrm{id}) \oplus \mathrm{id})$ restricts to give an injective homomorphism of $\mathfrak{A}^{(0)}$-modules $\widehat{X} : C^3_{(0)} \oplus C^1_{(0)} \to C^2_{(0)}$ (cf. [51, Lem. 7.1.5]). Further, since $\mathrm{Im}(\widehat{X})$ is equal to the image under X of $\mathrm{Im}(T) \oplus C^1_{(0)}$, the restriction of (4.23) then gives a canonical isomorphism

$$\begin{aligned} t^r \left[C^\bullet_{(0)}\right]^{-1}_{\mathfrak{A}^{(0)}} &= \left[C^1_{(0)}\right]_{\mathfrak{A}^{(0)}} \otimes_{\mathfrak{A}^{(0)}} \left[C^2_{(0)}\right]^{-1}_{\mathfrak{A}^{(0)}} \otimes_{\mathfrak{A}^{(0)}} \left(t^r \left[C^3_{(0)}\right]_{\mathfrak{A}^{(0)}}\right) \\ &= \left[C^1_{(0)}\right]_{\mathfrak{A}^{(0)}} \otimes_{\mathfrak{A}^{(0)}} \left[C^2_{(0)}\right]^{-1}_{\mathfrak{A}^{(0)}} \otimes_{\mathfrak{A}^{(0)}} [\mathrm{Im}(T)]_{\mathfrak{A}^{(0)}} \\ &\cong \left[\mathrm{Im}(T) \oplus C^1_{(0)}\right]_{\mathfrak{A}^{(0)}} \otimes_{\mathfrak{A}^{(0)}} \left[C^2_{(0)}\right]^{-1}_{\mathfrak{A}^{(0)}} \\ &\cong \left[\mathrm{Im}(\widehat{X})\right]_{\mathfrak{A}^{(0)}} \otimes_{\mathfrak{A}^{(0)}} \left[C^2_{(0)}\right]^{-1}_{\mathfrak{A}^{(0)}} \\ &\cong \left[\mathrm{Cok}(\widehat{X})\right]^{-1}_{\mathfrak{A}^{(0)}} \\ &= \left(\mathrm{Fitt}_{\mathfrak{A}^{(0)}}(\mathrm{Cok}(\widehat{X})), 0\right), \end{aligned}$$

where the last equality is by Lemma 2.1.6(ii). From this isomorphism it follows that $t^r \cdot \Xi(C^\bullet_{(0)})^{-1} = \mathrm{Fitt}_{\mathfrak{A}^{(0)}}(\mathrm{Cok}(\widehat{X}))$ and hence that $t^r \cdot \Xi(C^\bullet_{(0)})^{-1} \subseteq \mathrm{Ann}_{\mathfrak{A}^{(0)}}(\mathrm{Cok}(\widehat{X}))$. To deduce (4.22) it is thus enough to prove that $\mathrm{Ann}_{\mathfrak{A}^{(0)}}(\mathrm{Cok}(\widehat{X})) \subseteq \mathrm{Ann}_{\mathfrak{A}^{(0)}}(H^2_{(0)})$.

To do this we consider the finite $\mathfrak{A}^{(0)}$-module

$$N := \frac{\mathrm{Ker}(d^2_{(0)})}{\mathrm{Ker}(d^2_{(0)}) \cap \mathrm{Im}(\widehat{X})} \cong \frac{\mathrm{Ker}(d^2_{(0)})\,\mathrm{Im}(\widehat{X})}{\mathrm{Im}(\widehat{X})}.$$

Then the natural inclusion $N \subseteq \mathrm{Cok}(\widehat{X})$ implies $\mathrm{Ann}_{\mathfrak{A}^{(0)}}(\mathrm{Cok}(\widehat{X})) \subseteq \mathrm{Ann}_{\mathfrak{A}^{(0)}}(N)$. Also, the definition of $\widehat{X}$ ensures that $\mathrm{Im}(\widehat{X}|_{C^3_{(0)}}) \subseteq C^2_{(0)}$ and that the composite map $(d^2_{(0)} \circ \widehat{X})|_{C^3_{(0)}} = T$ is injective. The denominator $\mathrm{Ker}(d^2_{(0)}) \cap \mathrm{Im}(\widehat{X})$ in the definition of N is therefore contained in $\mathrm{Im}(\widehat{X}|_{C^1_{(0)}}) = \mathrm{Im}(d^1_{(0)})$. The cohomology group $H^2_{(0)} := \mathrm{Ker}(d^2_{(0)})/\,\mathrm{Im}(d^1_{(0)})$ is therefore a quotient of N and so $\mathrm{Ann}_{\mathfrak{A}^{(0)}}(\mathrm{Cok}(\widehat{X})) \subseteq \mathrm{Ann}_{\mathfrak{A}^{(0)}}(N) \subseteq \mathrm{Ann}_{\mathfrak{A}^{(0)}}(H^2_{(0)})$, as required to complete the proof of Proposition 4.3.7.

Having proved Proposition 4.3.7, we can now obtain (4.14) by simply substituting the results in the following lemma into the inclusion of Proposition 4.3.7.

Lemma 4.3.10.

(i) $I^{(0)} \cdot \mathrm{Ann}_{\mathfrak{A}}(H^3_{\mathrm{tor}}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^3_{(0)})$.

(ii) $(I^{(0)})^{n(C)} \cdot \mathrm{Ann}_{\mathfrak{A}}(H^2_{(0)}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2_{\mathrm{tor}})$ *where the integer* $n(C)$ *is equal to* 1 *if* H^1 *vanishes and is equal to* 2 *otherwise.*

Proof. Since $I^{(0)} = \mathfrak{A}^{(0)} \cap \mathfrak{A}$ the definition of $e^{(0)}$ combines with Remark 4.3.2 to imply that the module $H^3_{\mathrm{tf}} \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}$ is annihilated by (right multiplication by) $I^{(0)}$. The inclusion in claim (i) therefore follows from the natural exact sequence of $\mathfrak{A}$-modules $H^3_{\mathrm{tor}} \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \to H^3 \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \to H^3_{\mathrm{tf}} \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \to 0$ and the isomorphism $H^3 \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)} \cong H^3_{(0)}$ from Lemma 4.3.9.

Regarding claim (ii) we first note that for any $\mathfrak{A}$-module N and integer a the Tate cohomology group $\widehat{H}^a(G, N \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})$ is an $\mathfrak{A}$-module that is annihilated by $I^{(0)}$. To show this we write α for the projection $\mathfrak{A} \to \mathfrak{A}^{(0)}$. Then $N \otimes_{\mathbb{Z}_p} \mathfrak{A}$ is cohomologically trivial as a G-module (since $\mathfrak{A}$ is a direct factor of $\mathbb{Z}_p[G]$) and there is an exact sequence of $\mathbb{Z}_p[G]$-modules of the form

$$0 \to N \otimes_{\mathbb{Z}_p} \mathrm{Ker}(\alpha) \to N \otimes_{\mathbb{Z}_p} \mathfrak{A} \to N \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)} \to 0$$

(since $\mathfrak{A}^{(0)}$ is a free $\mathbb{Z}_p$-module). From the induced long exact sequence of Tate cohomology we therefore obtain a natural isomorphism

$$\widehat{H}^a(G, N \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}) \cong \widehat{H}^{a+1}(G, N \otimes_{\mathbb{Z}_p} \mathrm{Ker}(\alpha)).$$

Now the last module is certainly annihilated by $I^{(0)}$ if one has $xy = 0$ for every $x \in I^{(0)}$ and $y \in \mathrm{Ker}(\alpha)$. It is therefore enough to note that for each such x and y one has $xy = (xe^{(0)})y = x(e^{(0)}y) = x(\alpha(y)) = x.0 = 0$, as required.

In particular, since $I^{(0)}$ annihilates $H^2(G, H^1 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})$, the exact sequence (4.19) implies that $I^{(0)} \cdot \mathrm{Ann}_{\mathfrak{A}}(H^2_{(0)}) \subseteq \mathrm{Ann}_{\mathfrak{A}}((H^2 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G)$, respectively $\mathrm{Ann}_{\mathfrak{A}}(H^2_{(0)}) = \mathrm{Ann}_{\mathfrak{A}}((H^2 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G)$ if H^1 vanishes. Next we note that the definition of $e^{(0)}$ implies that the module $(H^2 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G$ is finite and hence also equal to $(H^2 \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G_{\mathrm{tor}} = (H^2_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G$. Further, the natural exact sequence

$$0 \to (H^2_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathrm{Ker}(\alpha))^G \to (H^2_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathfrak{A})^G \to (H^2_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G$$

combines with the isomorphism $H^2_{\mathrm{tor}} \cong (H^2_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathfrak{A})^G$ (coming from the fact that $\mathfrak{A}$ is a direct factor of $\mathbb{Z}_p[G]$) and the fact that $I^{(0)}$ annihilates $(H^2_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathrm{Ker}(\alpha))^G$ to imply that $I^{(0)} \cdot \mathrm{Ann}_{\mathfrak{A}}((H^2_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2_{\mathrm{tor}})$. Claim (ii) now follows directly upon combining all of the observations made in this paragraph and then recalling the definition of the integer $n(C)$. $\square$

4.3.5 The Proof of Theorem 4.3.3(ii)

Following Lemma 4.3.6 it is sufficient to prove the inclusion (4.15) for every admissible complex $C^\bullet$. We shall deduce (4.15) by substituting C^* for $C^\bullet$ in the inclusion (4.14) which was proved in the last subsection and then using Lemma 4.3.5. Indeed, to substitute C^* for $C^\bullet$ in (4.14) we already need to know that the complex C^* is admissible (as proved in Lemma 4.3.5(iii)) and that the idempotents $e^{(0)}_{C^*}$ and $e^{(0)}_{C^\bullet}$ coincide (as proved in Lemma 4.3.5(iv)). In fact we also know that $\Xi(C^*_{(0)}) = \Xi(C^\bullet_{(0)})^{-1}$ (by Lemma 4.3.5(v)) and that there are natural isomorphisms $H^i(C^*)_{\rm tor} \cong (H^{5-i}(C^\bullet)_{\rm tor})^\vee$ for $i = 2, 3$ (by Lemma 4.3.5(i)). Taking these facts into account and replacing $C^\bullet$ by C^* in (4.14) we obtain an inclusion

$$(I^{(0)})^{g(C^*)+n(C^*)}\,\mathrm{Ann}_{\mathfrak{A}}((H^2(C^\bullet)_{\rm tor})^\vee)^{g(C^*)}\cdot\Xi(C^\bullet_{(0)}) \subseteq \mathrm{Ann}_{\mathfrak{A}}((H^3(C^\bullet)_{\rm tor})^\vee).$$

To deduce the required inclusion (4.15) from this we need only note that for every finite $\mathfrak{A}$-module M one has $\mathrm{Ann}_{\mathfrak{A}}(M^\vee) = \mathrm{Ann}_{\mathfrak{A}}(M)$.

4.3.6 The Proof of Theorem 4.3.3(iii)

At this stage we have proved Theorem 4.3.3(i) and (ii) and so, to complete the proof of Theorem 4.3.3, it only remains to prove claim (iii). In this subsection we shall prove claim (iii) by specialising arguments used by Andrew Parker in his thesis (cf. [48, Thm. 4.3.1]).

At the outset we fix a strongly admissible complex of $\mathfrak{A}$-modules $C^\bullet$ and assume, as per Theorem 4.3.3(iii), that either $H^2(C^\bullet)$ is $\mathbb{Z}_p$-free or $\mathfrak{A}^{(0)}$ is relatively Gorenstein over $\mathbb{Z}_p$ (which we recall means that $\mathfrak{A}^{(0)*}$ is isomorphic to $\mathfrak{A}^{(0)}$ as an $\mathfrak{A}^{(0)}$-module).

We start by recording a useful technical result.

Lemma 4.3.11. *Let $C^\bullet$ be a strongly admissible complex of $\mathfrak{A}$-modules for which either $H^2(C^\bullet)$ is $\mathbb{Z}_p$-free or $\mathfrak{A}^{(0)}$ is relatively Gorenstein over $\mathbb{Z}_p$.*

(i) $\mathrm{Fitt}_{\mathfrak{A}^{(0)}}(H^3(C^\bullet_{(0)})) = \mathrm{Fitt}_{\mathfrak{A}}(H^3(C^\bullet)).$

(ii) $\mathrm{Fitt}_{\mathfrak{A}^{(0)}}(H^2(C^\bullet_{(0)})^\vee) = \mathrm{Fitt}_{\mathfrak{A}}((H^2(C^\bullet)_{\mathrm{tor}})^\vee)e^{(0)}.$

Proof. Both complexes $C^\bullet$ and $C^\bullet_{(0)}$ are acyclic outside degrees 2 and 3 and, since $H^1(C^\bullet) = 0$, Lemma 4.3.9 implies that there are canonical isomorphisms of finite $\mathfrak{A}^{(0)}$-modules

$$H^n(C^\bullet_{(0)}) \cong \begin{cases} (H^2(C^\bullet) \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G, & n = 2 \\ H^3(C^\bullet) \otimes_{\mathfrak{A}} \mathfrak{A}^{(0)}, & n = 3. \end{cases} \tag{4.24}$$

Further, in this case Remark 4.3.2 implies that $e^{(0)}$ is equal to the sum of all primitive idempotents of A which annihilate $H^3(C^\bullet) \otimes_{\mathfrak{A}} A$. The equality of claim (i) therefore follows from Lemma 2.1.7(ii) with $M = H^3(C^\bullet)$.

Next we note that since $H^2(C^\bullet_{(0)})$ is finite and $\mathfrak{A}^{(0)}$ is $\mathbb{Z}_p$-free the isomorphism (4.24) implies that $H^2(C^\bullet_{(0)})$ is isomorphic to $(H^2(C^\bullet) \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G = (H^2(C^\bullet) \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G_{\mathrm{tor}} = (H^2(C^\bullet)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^G$. In particular, if $H^2(C^\bullet)$ is $\mathbb{Z}_p$-free, then $H^2(C^\bullet_{(0)})$ vanishes and so in this case the equality of claim (ii) is obvious.

On the other hand, if $\mathfrak{A}^{(0)*}$ is isomorphic to $\mathfrak{A}^{(0)}$ as an $\mathfrak{A}^{(0)}$-module, then there are isomorphisms of $\mathbb{Z}_p[G]$-modules

$$(H^2(C^\bullet)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)})^\vee \cong (H^2(C^\bullet)_{\mathrm{tor}})^\vee \otimes_{\mathbb{Z}_p} (\mathfrak{A}^{(0)})^* \cong (H^2(C^\bullet)_{\mathrm{tor}})^\vee \otimes_{\mathbb{Z}_p} \mathfrak{A}^{(0)}$$

(cf. [48, Lem. 4.3.6]). From (4.24) we therefore obtain isomorphisms of $\mathfrak{A}^{(0)}$-modules of the form

$$H^2(C^\bullet_{(0)})^\vee \cong ((H^2(C^\bullet)\otimes_{\mathbb{Z}_p}\mathfrak{A}^{(0)})^G)^\vee = ((H^2(C^\bullet)_{\mathrm{tor}}\otimes_{\mathbb{Z}_p}\mathfrak{A}^{(0)})^G)^\vee$$
$$\cong ((H^2(C^\bullet)_{\mathrm{tor}}\otimes_{\mathbb{Z}_p}\mathfrak{A}^{(0)})^\vee)_G \cong ((H^2(C^\bullet)_{\mathrm{tor}})^\vee\otimes_{\mathbb{Z}_p}\mathfrak{A}^{(0)})_G \cong (H^2(C^\bullet)_{\mathrm{tor}})^\vee\otimes_{\mathfrak{A}}\mathfrak{A}^{(0)}$$

Given this isomorphism, the equality of claim (ii) follows from Lemma 2.1.7(i). $\square$

We now return to the proof of Theorem 4.3.3(iii). If firstly $H^2(C^\bullet)$ is $\mathbb{Z}_p$-free, then Lemma 4.3.11(ii) implies that $C^\bullet_{(0)}$ is acyclic outside degree 3 and so is isomorphic in $\mathcal{D}^p(\mathfrak{A}^{(0)})$ to $H^3(C^\bullet_{(0)})[-3]$. By Lemma 2.1.6 we therefore obtain a canonical isomorphism of graded $\mathfrak{A}^{(0)}$-modules $\left[C^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}} \cong (\mathrm{Fitt}_{\mathfrak{A}^{(0)}}(H^3(C^\bullet_{(0)})),0)$. By Lemma 4.3.11(i) this last isomorphism implies that of Theorem 4.3.3(iii) in the case that $H^2(C^\bullet)$ is $\mathbb{Z}_p$-free.

To deal with the general case we note that since $C^\bullet_{(0)}$ is acyclic outside degrees 2 and 3 it is quasi-isomorphic to the complex $\mathrm{Cok}(d^1_{(0)}) \xrightarrow{d^2_{(0)}} C^3\otimes_{\mathfrak{A}}\mathfrak{A}^{(0)}$ where the first term is placed in degree 2 and we have used the same notation as in (4.20). We note that $C^3_{(0)}$ is a finitely generated free $\mathfrak{A}^{(0)}$-module, that $\mathrm{Cok}(d^1_{(0)})$ is a finitely generated $\mathfrak{A}^{(0)}$-module which has projective dimension at most one and that there is a natural 2-extension of $\mathfrak{A}^{(0)}$-modules of the form

$$0\to H^2(C^\bullet_{(0)})\to \mathrm{Cok}(d^1_{(0)})\to C^3_{(0)}\to H^3(C^\bullet_{(0)})\to 0. \tag{4.25}$$

We now choose a finitely generated free $\mathfrak{A}^{(0)}$-module F and an integer i for which there exists a surjection $F/p^iF \twoheadrightarrow H^3(C^\bullet_{(0)})$. Let N denote the kernel of this map. Since the projective dimension of F/p^iF is at most one the group $\mathrm{Ext}^2_{\mathfrak{A}^{(0)}}(F/p^iF, H^2(C^\bullet_{(0)}))$ vanishes (by [64, Lemma 4.1.6]) and so there exists a natural surjection $\delta : \mathrm{Ext}^1_{\mathfrak{A}^{(0)}}(N, H^2(C^\bullet_{(0)})) \twoheadrightarrow \mathrm{Ext}^2_{\mathfrak{A}^{(0)}}(H^3(C^\bullet_{(0)}), H^2(C^\bullet_{(0)}))$. We choose any Yoneda extension

$$0 \to H^2(C^\bullet_{(0)}) \to M \to N \to 0 \tag{4.26}$$

which represents a pre-image under δ of the class of the extension (4.25). Then, by the description of δ in terms of Yoneda extensions, we may splice (4.26) with the extension $0 \to N \to F/p^iF \to H^3(C^\bullet_{(0)}) \to 0$ to obtain a 2-extension of $\mathfrak{A}^{(0)}$-modules

$$0 \to H^2(C^\bullet_{(0)}) \to M \to F/p^iF \to H^3(C^\bullet_{(0)}) \to 0 \tag{4.27}$$

that is Yoneda equivalent to (4.25). The complex

$$M \to F/p^iF \tag{4.28}$$

of $\mathfrak{A}^{(0)}$-modules in degrees 2 and 3 is therefore isomorphic to $C^\bullet_{(0)}$ in $\mathcal{D}^p(\mathfrak{A})$ via an isomorphism that induces the identity on cohomology groups in degrees 2 and 3. We note that, since N, $H^2(C^\bullet_{(0)})$ and $H^3(C^\bullet_{(0)})$ are all finite, so is M. Further, since (4.28) is a perfect complex and F/p^iF has projective dimension at most one the module M also has projective dimension at most one. The quasi-isomorphism between (4.28) and $C^\bullet_{(0)}$ therefore induces a canonical isomorphism of determinant modules $\left[C^\bullet_{(0)}\right]_{\mathfrak{A}^{(0)}} \cong [F/p^iF[-3]]_{\mathfrak{A}^{(0)}}[M[-2]]_{\mathfrak{A}^{(0)}}$ which, by Lemma 2.1.6, in turn induces an equality

$$\Xi(C^\bullet_{(0)}) = \mathrm{Fitt}_{\mathfrak{A}^{(0)}}(F/p^iF) \cdot \mathrm{Fitt}_{\mathfrak{A}^{(0)}}(M)^{-1}.$$

Thus, if we now assume that $\mathfrak{A}^{(0)}$ is relatively Gorenstein over $\mathbb{Z}_p$, then the result of

[14, Lem. 5] applies to the sequence (4.27) to give an equality

$$\begin{aligned}\mathrm{Fitt}_{\mathfrak{A}^{(0)}}(H^2(C_{(0)})^\vee)\cdot\Xi(C^\bullet_{(0)}) &= \mathrm{Fitt}_{\mathfrak{A}^{(0)}}(H^2(C_{(0)})^\vee)\,\mathrm{Fitt}_{\mathfrak{A}^{(0)}}(F/p^iF)\,\mathrm{Fitt}_{\mathfrak{A}^{(0)}}(M)^{-1}\\ &= \mathrm{Fitt}_{\mathfrak{A}^{(0)}}(H^3(C_{(0)}))^{-1}.\end{aligned}$$

The equality of Theorem 4.3.3(iii) now follows by simply substituting into this equality the equalities of Lemma 4.3.11.

This completes our proof of Theorem 4.3.3(iii) in the case that $\mathfrak{A}^{(0)}$ is relatively Gorenstein over $\mathbb{Z}_p$, and hence our proof of all parts of Theorem 4.3.3.

4.4 The Cohomology of $R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z}_p(r))$

To prove Theorems 4.1.1 and 4.1.5 we shall apply the relevant parts of Theorem 4.3.3 in the context of the leading term formulas in Theorem 4.2.1 and then use an explicit description of the cohomology modules of complexes of the form $R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z}_p(r))$. In this section we shall record the necessary descriptions of these cohomology modules. The results in this section are in principle certainly well known but for convenience we prefer to record them (they will in fact also be useful in Chapter 5). For completeness we will also give proofs of the results recorded here.

Throughout we fix notation as in Theorem 4.2.1. In particular, K is a finite abelian CM extension of a totally real field k, $G = \mathrm{Gal}(K/k)$ and r is a *non-zero* integer. We recall also that S denotes the (finite) set of places of k consisting of all places which ramify in K/k (which therefore includes all archimedean places) and all places lying above p and that $\mathcal{G}_{K,S}$ denotes the galois group over K of the maximal abelian pro-p extension of K inside $\overline{K}$ that is unramified outside of $S(K)$.

We use the $G\times\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module $Y_{K,r} = \bigoplus_{K\hookrightarrow\mathbb{C}}(2\pi i)^r\mathbb{Z}$ introduced in §2.2.1 and recall that $Y_{K,r}^{+}$ denotes the G-submodule of $Y_{K,r}$ comprising elements invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

For convenience we often write $C_r^{\bullet}$ in place of $R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$.

4.4.1 CM Extensions

Proposition 4.4.1.

(*i.*) $R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ *is an admissible complex of* $\mathbb{Z}_p[G]$*-modules.*

(*ii.*) *There is a canonical short exact sequence of* $\mathbb{Z}_p[G]$*-modules*

$$0 \to Y_{K,r}^{+} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H_c^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to 0.$$

If Schneider's conjecture (Conjecture 3.2.9) holds for K *at* r *and* p *then this exact sequence induces a canonical isomorphism*

$$H_c^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \cong Y_{K,r}^{+} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

(*iii.*) *There is a canonical isomorphism of* $\mathbb{Z}_p[G]$*-modules* $H_c^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(1)) \cong \mathcal{G}_{K,S}$*, where* $\mathcal{G}_{K,S}$ *is regarded as a* $\mathbb{Z}_p[G]$*-module by the natural conjugation action of* G *on* $\mathcal{G}_{K,S}$*. With respect to canonical Kummer theory isomorphisms* $H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(1)) \cong \mathcal{O}_{K,S}^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ *and* $P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(1)) \cong \prod_{w \in S(K)} K_w^{\times} \widehat{\otimes} \mathbb{Z}_p$ *the map* ρ_1^1 *from* $H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(1))$ *to* $P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(1))$ *identifies with the natural diagonal map*

$$\lambda_{K,S} : \mathcal{O}_{K,S}^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \prod_{w \in S(K)} K_w^{\times} \widehat{\otimes} \mathbb{Z}_p.$$

(*iv.*) *There is a canonical isomorphism of* $\mathbb{Z}_p[G]$*-modules*

$$H^3_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \cong \mathbb{Z}_p(r-1)_{G_K}.$$

Proof. One knows that the complex $C_r^\bullet$ belongs to $\mathcal{D}^p(\mathbb{Z}_p[G])$ by, for example, [10, Prop. 1.20(a)] and that the Euler characteristic of $\mathbb{Q}_p[G] \otimes_{\mathbb{Z}_p[G]} C_r^\bullet = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} C_r^\bullet \cong R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ in $K_0(\mathbb{Q}_p[G])$ vanishes by, for example, [30, Prop. 2.1.3]. The fact that the cohomological dimensions of $G_{K,S}$ and of each G_v is 2 (cf. [45, Thms. 7.1.8 & 10.9.3]) also combines with the long exact sequence of cohomology of the exact triangle (3.1) to imply that $H^i(C_r^\bullet)$ vanishes for $i \notin \{0,1,2,3\}$ and so, recalling Lemma 3.1.6, it follows that $C_r^\bullet$ is acyclic outside degrees $1, 2$ and 3. To prove that $C_r^\bullet$ is admissible it therefore suffices to prove that $H^1(C_r^\bullet) = H^1_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is $\mathbb{Z}_p$-free. But the long exact sequence induced by the natural short exact sequence $0 \to \mathbb{Z}_p(r) \to \mathbb{Q}_p(r) \to \mathbb{Q}_p/\mathbb{Z}_p(r) \to 0$ induces an isomorphism of $H^1_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))_{\mathrm{tor}}$ with $H^0_c(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(r))$ and the latter module vanishes as a consequence of Lemma 3.1.6. This proves claim (i).

Since r is non-zero the group $H^0(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ vanishes and so the long exact cohomology sequence of the exact triangle (3.1) restricts to give an exact sequence

$$0 \to P^0(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to H^1_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \xrightarrow{\rho_r^1} P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)).$$

Since $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is by definition equal to $\ker(\rho_r^1)$ to derive the short exact sequence in claim (ii) it is therefore enough to prove that $P^0(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ is naturally isomorphic to $Y_{K,r}^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Recalling the isomorphism $P^0(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \cong (\mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K)^+$ from (the proof of) Lemma 4.2.2(i) it is thus enough to note that there is a natural isomorphism of $\mathbb{Z}_p[G]$-modules

$$\gamma^+ : (\mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K)^+ \cong Y_{K,r}^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p. \tag{4.29}$$

Indeed if we regard K as a subfield of $\mathbb{C}$, then $\mathrm{Ind}_k^{\mathbb{Q}} \mathbb{Z}_p(r)_K$ can be identified with $\bigoplus_{K\hookrightarrow\mathbb{C}} \mathbb{Z}_p(r)$ and one obtains γ^+ as the restriction of the isomorphism

$$\bigoplus_{K\hookrightarrow\mathbb{C}} \mathbb{Z}_p(r) \cong (\bigoplus_{K\hookrightarrow\mathbb{C}} (2\pi i)^r\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p = Y_{K,r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

obtained by "swapping" the topological generator $(\exp(2\pi i/p^n)^{\otimes r})_{n\geq 0}$ of $\mathbb{Z}_p(r)$ with the generator $(2\pi i)^r$ of the lattice $(2\pi i)^r\mathbb{Z}$. The second assertion of claim (ii) follows immediately from the short exact sequence since if Schneider's conjecture holds for K at r and p, then Lemma 3.2.10(ii) implies that the group $\text{Ш}^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ vanishes.

To prove claims (iii) and (iv) we recall that the Artin-Verdier duality theorem implies that if $a \in \{2,3\}$, then there is a canonical isomorphism of $\mathbb{Z}_p[G]$-modules $H_c^a(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \cong H^{3-a}(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee$ (cf. [43, Chap. III, Cor. 3.4]). In fact, since the action of G_K on $\mathbb{Q}_p/\mathbb{Z}_p(1-r)$ is via the quotient group $G_{K,S}$ one has $H^0(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r)) = (\mathbb{Q}_p/\mathbb{Z}_p(1-r))^{G_K}$ so $H^0(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee = ((\mathbb{Q}_p/\mathbb{Z}_p(1-r))^{G_K})^\vee \cong ((\mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee)_{G_K} \cong \mathbb{Z}_p(r-1)_{G_K}$ and so claim (iv) follows immediately from the Artin-Verdier duality isomorphism with $a = 3$.

In a similar way, the first assertion of claim (iii) follows from the Artin-Verdier duality isomorphism with $a = 2$ and $r = 1$ and the fact that there is a natural isomorphism $H^1(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \mathcal{G}_{K,S}$ (for an explicit description of the latter isomorphism see, for example, the proof of [14, Lem. 3]). Finally we note that the description of ρ_1^1 as the natural diagonal homomorphism $\lambda_{K,S}$ is also well-known (see, for example, the discussion of [14, p.177]). $\square$

In the next two subsections we record some useful consequences of Proposition 4.4.1 for the "plus part" and "minus part" of the complex $C_r^\bullet$. To do this we regard the non-zero integer r (and prime p) as fixed and set $\mathfrak{A}^\pm := \mathbb{Z}_p[G]e_r^\pm$, $A^\pm := \mathfrak{A}^\pm \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p[G]e_r^\pm$ and $R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))^\pm = C_r^{\bullet\pm} := C_r^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}^\pm \in \mathcal{D}^p(\mathfrak{A}^\pm)$. We note in particular that since $\mathfrak{A}^\pm$ is a direct factor of $\mathbb{Z}_p[G]$ in each degree n there are natural identifications $H^n(C_r^{\bullet\pm}) = H^n(C_r^\bullet e_r^\pm) = H^n(C_r^\bullet)e_r^\pm$.

4.4.2 The "Plus-part"

Proposition 4.4.2.

(i) $R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))^+$ *is an admissible complex of* $\mathfrak{A}^+$*-modules.*

(ii) There are canonical isomorphisms of $\mathfrak{A}^+$*-modules*

$$H^i_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))\, e_r^+ \cong \begin{cases} Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) & i = 1, \\ \mathbb{Z}_p(r-1)_{G_K} & i = 3. \end{cases}$$

(iii) Schneider's conjecture (Conjecture 3.2.9) holds for K at r and p if and only if the complex $R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))^+$ is strongly admissible.

(iv) If $r \notin \{0, 1\}$ and Schneider's conjecture holds for K at r and p, then the idempotent $e_r^{(0)}$ (defined in Theorem 4.2.1) is equal to e_r^+.

(v) If $r = 1$ and Leopoldt's conjecture holds for K at p, then the idempotent $e_1^{(0)}$ (defined in Theorem 4.2.1) is equal to $e_1^+ - e_G$.

Proof. Claim (i) follows directly from Proposition 4.4.1(i). Since the definition of the G action on $Y_{K,r}^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p$ makes it clear that $(Y_{K,r}^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p)e_r^+ = 0$, from Proposition

4.4.1(ii) and (iv) we also obtain canonical isomorphisms of $\mathfrak{A}^+$-modules

$$H^i_c\left(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)\right) e_r^+ \cong \begin{cases} Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^+ & i = 1, \\ \left(\mathbb{Z}_p(r-1)_{G_K}\right) e_r^+ & i = 3. \end{cases}$$

To prove claim (ii) it therefore suffices to prove that both $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^+ = Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ and $(\mathbb{Z}_p(r-1)_{G_K})e_r^+ = \mathbb{Z}_p(r-1)_{G_K}$. The second equality is clear because complex conjugation acts on $\mathbb{Z}_p(r-1)$ as multiplication by $(-1)^{r-1}$ (and so e_r^+ acts as the identity on $\mathbb{Z}_p(r-1)_{G_K}$). The first equality follows immediately from claim (i), respectively (iii), of Lemma 3.2.10 if $r < 0$, respectively $r > 1$, and so it suffices to prove it in the case $r = 1$. In this case Proposition 4.4.1(iii) identifies $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(1)) = \ker(\rho_1^1)$ with $\ker(\lambda_{K,S})$. But by [63, Thm. 4.12] one knows that the module $\ker(\lambda_{K,S})e_1^+ = \ker(\lambda_{K,S})e^+$ is finite and so the group $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(1))e_1^+$ does indeed vanish, being both finite and torsion-free (by the argument in the proof of Lemma 3.2.10). This proves claim (ii).

Next we note that, since $C_r^{\bullet +}$ is admissible, it is strongly admissible precisely when the group $H^1(C_r^{\bullet +}) \cong Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(1))$ vanishes. Claim (iii) therefore follows immediately from Lemma 3.2.10(ii).

Regarding claims (iv) and (v) we note first that the definition of the idempotent $e_r^{(0)}$ in the statement of Theorem 4.2.1 implies that it is equal to the idempotent $e_{C_r^\bullet}^{(0)}$ in the terminology introduced in §4.3.1. But the equality $e_r^{(0)} = e_r^+ e_r^{(0)}$ of Theorem 4.2.1(i) then implies that $e_{C_r^\bullet}^{(0)} = e_{C_r^{\bullet +}}^{(0)}$ and so it suffices to prove that $e_{C_r^{\bullet +}}^{(0)} = e_r^+$, respectively $e_{C_1^{\bullet +}}^{(0)} = e_1^+ - e_G$ if $r \notin \{0, 1\}$, respectively $r = 1$, and Schneider's Conjecture is valid for K at r and p (we recall from Remark 3.2.11 that if $r = 1$, then Schneider's Conjecture for K at r and p is equivalent to Leopoldt's Conjecture for K at p). Now if Schneider's Conjecture is valid, then claim (iii) implies that $C_r^{\bullet +}$ is acyclic outside degrees 2 and 3 and so Remark 4.3.2 implies that $e^{(0)}_{C^{\bullet +}_r}$ is equal to the sum of all primitive idempotents of A^+ which annihilate $H^3(C^{\bullet +}_r)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. But the isomorphisms of claim (ii) imply that $H^3(C^{\bullet +}_r)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ vanishes, respectively is isomorphic to $\mathbb{Q}_p$ (with trivial G-action) if $r\notin\{0,1\}$, respectively $r=1$. It is therefore clear that $e^{(0)}_{C^{\bullet +}_r}=e^+_r$, respectively $e^{(0)}_{C^{\bullet +}_1}=e^+_1-e_G$ if $r\notin\{0,1\}$, respectively $r=1$, as required to complete the proof of claims (iv) and (v). $\square$

If $r=1$, then e^+_r is equal to the idempotent $e^+=\frac{1}{2}(1+\tau)$. In this case exactly the same argument as in the proof of Proposition 4.4.2 therefore proves the following result for abelian extensions of totally real fields.

Corollary 4.4.3. *Let L/k be a finite abelian extension of totally real fields.*

(i) $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ is an admissible complex of $\mathbb{Z}_p[G_{L/k}]$-modules.

(ii) There are canonical isomorphisms of $\mathbb{Z}_p[G_{L/k}]$-modules

$$H^i_c(\mathcal{O}_{L,S},\mathbb{Z}_p(1))\cong\begin{cases}\mathcal{G}_{L,S} & \text{if } i=2,\\ \mathbb{Z}_p & \text{if } i=3.\end{cases}$$

(iii) Set $C^\bullet := R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$. If Leopoldt's conjecture holds for L at p, then $C^\bullet$ is a strongly admissible complex of $\mathbb{Z}_p[G_{L/k}]$-modules. The idempotents $e^{(0)}_1$ (defined in Theorem 4.2.1) and $e^{(0)}_{C^\bullet}$ (defined in §4.3.1) are both equal to $1-e_{G_{L/k}}$ and so the ideal $I^{(0)}_{C^\bullet}$ defined in §4.3.1 is equal to the kernel $I_{G_{L/k}}$ of the augmentation homomorphism $\mathbb{Z}_p[G_{L/k}]\twoheadrightarrow\mathbb{Z}_p$ which sends each element of $G_{L/k}$ to 1.

4.4.3 The "Minus-part"

Proposition 4.4.4. *Assume that the integer r is strictly greater than one.*

(i.) $R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))^-$ *is a perfect complex of* $\mathfrak{A}^-$*-modules that is acyclic outside of degrees* 1 *and* 2*.*

(ii.) There is a canonical isomorphism of $\mathfrak{A}^-$*-modules*

$$H^1_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^- \cong Y^+_{K,r} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

(iii.) The Euler characteristic of the complex $R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))^-$ *in* $K_0(A^-)$ *is zero.*

Proof. Proposition 4.4.1(i) implies claim (iii) directly and also that $R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))^-$ is a perfect complex of $\mathfrak{A}^-$-modules that is acyclic outside of degrees 1, 2 and 3. To deduce claim (i) it therefore suffices to prove that $H^3_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))^-$ vanishes. But this is true because Proposition 4.4.1(i) gives an isomorphism $H^3_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \cong \mathbb{Z}_p(r-1)_{G_K}$ and the latter module is annihilated by e_r^- (see the proof of Proposition 4.4.2(ii)).

Regarding claim (ii) we combine the exact sequence of Proposition 4.4.1(ii) with the fact that $Ш^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$ vanishes (by Lemma 3.2.10(iii)) to obtain a natural isomorphism of $\mathfrak{A}^-$-modules $H^1_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^- \cong (Y^+_{K,r} \otimes_{\mathbb{Z}} \mathbb{Z}_p)e_r^-$. It is therefore enough to note that the explicit description of the action of $\tau(\in G)$ on $Y_{K,r}$ implies that e_r^- acts as the identity on $Y^+_{K,r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. □

4.5 The Proof of Theorem 4.1.1

We are at last ready to prove Theorems 4.1.1 and 4.1.5.

In this section we deal with Theorem 4.1.1 and so fix a finite abelian extension L/k of totally real fields. We recall that $S = S_{L/k}$ denotes the (finite) set consisting of all archimedean places of k, all places which ramify in L/k and all places which lie above p. We also assume (as in Theorem 4.1.1) that there exists a CM abelian extension K of k which contains L and is such that $\mu(K,p)$ vanishes, K/k is unramified outside of $S(L)$ and L is the maximal totally real subfield of K. We assume that L validates Leopoldt's conjecture at p and note that this implies K also validates Leopoldt's conjecture at p (see, for example, the argument at the end of the first paragraph of the proof of Proposition 4.4.2).

We set $G := G_{K/k}$, $\mathfrak{A} := \mathbb{Z}_p[G]e_1^+$, $A := \mathfrak{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $C^\bullet := R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(1))e_1^+$. We write $e^{(0)}$ for the sum of all primitive idempotents of A which annihilate $H^2(C^\bullet)\otimes_{\mathfrak{A}} A$ and recall that by Proposition 4.4.2(v) this idempotent is equal to $e_1^+(1-e_G)$. The ideal $\mathfrak{A}^{(0)}_{C^\bullet}$ introduced in §4.3.1 is therefore equal to the ideal $\mathfrak{A}^{(0)}_1(1-e_G)$ which occurs in Theorem 4.2.1(ii). In terms of the notation used in Theorem 4.3.3, it then also follows that the complex $(C^\bullet)_{(0)}$ is equal to the complex $C^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathfrak{A}^{(0)}_1(1-e_G)$ which occurs in Theorem 4.2.1(ii).

Now from Proposition 4.4.2(ii) and (iii) we know that the complex $C^\bullet$ is a strongly admissible complex of $\mathfrak{A}$-modules and that the module $H^3(C^\bullet) \cong \mathbb{Z}_p$ needs only one generator as an $\mathfrak{A}$-module and is $\mathbb{Z}_p$-free. Thus in terms of the notation used in Theorem 4.2.1(ii) we have $g(C^\bullet) = 1$, $n(C^\bullet) = 0$ and the ideal $I^{(0)}_{C^\bullet}$ is equal to $\mathfrak{A} \cap \mathfrak{A}(1-e_G) = I_G e_1^+ \cong I_{G_{L/k}}$. We may therefore apply both Theorem 4.3.3(i) and

Theorem 4.2.1(ii) in this context to obtain an inclusion

$$\begin{aligned}
L^*_{p,S}(1)I^2_{G_{L/k}} &= L^*_{p,S}(1)I_{C^\bullet}^{g(C^\bullet)+n(C^\bullet)}\operatorname{Ann}_{\mathfrak{A}}(H^3(C^\bullet)_{\rm tor})^{g(C^\bullet)}\\
&= I_{C^\bullet}^{g(C^\bullet)+n(C^\bullet)}\operatorname{Ann}_{\mathfrak{A}}(H^3(C^\bullet)_{\rm tor})^{g(C^\bullet)}[C^\bullet\otimes_{\mathfrak{A}}\mathfrak{A}(1-e_G)]^{-1}_{\mathfrak{A}(1-e_G)}\\
&\subseteq \operatorname{Ann}_{\mathfrak{A}}(H^2(C^\bullet)_{\rm tor}) = \operatorname{Ann}_{\mathfrak{A}}(e_1^+(\mathcal{G}_{K,S})_{\rm tor}) = \operatorname{Ann}_{\mathfrak{A}}((\mathcal{G}_{L,S})_{\rm tor})
\end{aligned}$$

where the penultimate equality follows from the isomorphism of Proposition 4.4.1(iii) and the last from that of Lemma 2.2.5 (with $r=1$). To derive the inclusions of (4.1) from this it suffices to note that since L validates Leopoldt's Conjecture at p the extension $M^p_S(L)/L^\infty$ is finite and so $\mathcal{H}_{L,S}=(\mathcal{G}_{L,S})_{\rm tor}$ and that there is natural surjective homomorphism $\mathcal{H}_{L,S}=\operatorname{Gal}(M^p_S(L)/L^\infty)\to\operatorname{Gal}(H(L)/H(L)\cap L^\infty)\otimes\mathbb{Z}_p=\operatorname{Cl}_{(\infty)}(L)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ and hence an inclusion $\operatorname{Ann}_{\mathfrak{A}}(\mathcal{H}_{L,S})\subseteq\operatorname{Ann}_{\mathbb{Z}[G]}(\operatorname{Cl}_{(\infty)}(L))\otimes_{\mathbb{Z}}\mathbb{Z}_p$.

In an entirely similar way, if the Sylow p-subgroup of $G_{L/k}$ (and hence also of G) is cyclic, then $\mathbb{Z}_p[G]e^{(0)}$ is relatively Gorenstein over $\mathbb{Z}_p$ and so we can combine the equality of Theorem 4.3.3(iii) with the leading term formula of Theorem 4.2.1(ii) and the result of Lemma 2.1.8 to obtain an equality

$$\begin{aligned}
L^*_{p,S}(1)I_{G_{L/k}} &= L^*_{p,S}(1)\operatorname{Fitt}_{\mathbb{Z}_p[G_{L/k}]}(\mathbb{Z}_p) = L^*_{p,S}(1)\operatorname{Fitt}_{\mathfrak{A}}(H^3(C^\bullet))\\
&= \operatorname{Fitt}_{\mathfrak{A}}((H^2(C^\bullet)_{\rm tor})^\vee) = \operatorname{Fitt}_{\mathfrak{A}}(H^2(C^\bullet)_{\rm tor}) = \operatorname{Fitt}_{\mathbb{Z}_p[G]}(\mathcal{H}_{L,S})
\end{aligned}$$

To deduce the remaining inclusions (4.2) and (4.4) of Theorem 4.1.1 from the inclusion (4.1) and equality (4.3) respectively it clearly suffices to prove that one has an inclusion $I_{G_{L/k}}\operatorname{Ann}_{\mathbb{Z}[G_{L/k}]}(\operatorname{Cl}_{(\infty)}(L))\otimes_{\mathbb{Z}}\mathbb{Z}_p\subseteq\operatorname{Ann}_{\mathbb{Z}[G_{L/k}]}(\operatorname{Cl}(L))\otimes_{\mathbb{Z}}\mathbb{Z}_p$. But this is an easy consequence of the natural exact sequence

$$0 \to \mathrm{Cl}_{(\infty)}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{Cl}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{Gal}((H(L) \cap L^{\infty})/L) \to 0$$

and the fact that the $\mathbb{Z}_p[G_{L/k}]$-module $\mathrm{Gal}((H(L) \cap L^{\infty})/L)$ is isomorphic to a quotient of $\mathrm{Gal}(L^{\infty}/L) \cong \mathbb{Z}_p$ and so is annihilated by any element of $I_{G_{L/k}}$.

This completes the proof of Theorem 4.1.1.

4.6 The Proof of Theorem 4.1.5

Since claim (iii) of Theorem 4.1.5 follows directly from Lemma 3.2.7 we shall focus on claims (i) and (ii). To do this we fix a CM abelian extension K of a totally real field k and set $G := \mathrm{Gal}(K/k)$. We assume that K contains a primitive p^{th} root of unity and that the μ-invariant $\mu(K, p)$ vanishes.

For each integer r we set $\mathfrak{A}_r^+ := \mathbb{Z}_p[G]e_r^+$ and $A_r^+ := \mathfrak{A}_r^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We also make use of the twisted p-adic L-functions $\mathfrak{L}_{p,S}(\cdot)$ from Definition 2.2.8.

Proposition 4.6.1. *Let r be an integer not equal to 0 or 1 such that Schneider's conjecture holds for K at r and p. Then one has*

$$\mathfrak{L}_{p,S}(r)\,\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_K}) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}\left(Ш(\mathbb{Z}_p(r)_K)\right) e_r^+.$$

Further, if r is strictly greater than one and the Sylow p-subgroup of G is cyclic, then the inclusion is still valid if one replaces the term $\mathrm{Ann}_{\mathbb{Z}_p[G]}\left(Ш(\mathbb{Z}_p(r)_K)\right)$ by $\mathrm{Fitt}_{\mathbb{Z}_p[G]}\left(Ш(\mathbb{Z}_p(r)_K)\right)$.

Proof. We set $C_r^{\bullet} := R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ and $C_r^{\bullet +} := R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))^+$ (using the notation introduced just prior to Proposition 4.4.2).

We first recall from Proposition 4.4.2(iii) and (iv) that if Schneider's conjecture holds for K at p and r, then $C_r^{\bullet +}$ is strongly admissible and both the idempotents $e_r^{(0)}$ (defined in Theorem 4.2.1) and $e_{C_r^\bullet}^{(0)}$ (used in Theorem 4.3.3) are equal to e_r^+. The algebra $\mathfrak{A}_{C_r^\bullet}^{(0)}$ which occurs in Theorem 4.3.3 is therefore equal to $\mathfrak{A}_r^+ = \mathbb{Z}_p[G]e_r^+$ and so, since this algebra is relatively Gorenstein over $\mathbb{Z}_p$, we may apply Theorem 4.3.3(iii) with $C^\bullet = C_r^\bullet$. Taking account of the leading term formula of Theorem 4.2.1 (noting that the order $\mathfrak{A}_r^{(0)}$ defined there is equal to $\mathfrak{A}_r^+$) and of the explicit description of $H^3(C_r^\bullet)$ given in Proposition 4.4.2(ii) we thereby obtain an equality

$$\mathfrak{L}_{p,S}(r)\,\mathrm{Fitt}_{\mathfrak{A}_r^+}(\mathbb{Z}_p(r-1)_{G_K}) = \mathrm{Fitt}_{\mathfrak{A}_r^+}((H_c^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^+)^\vee). \tag{4.30}$$

Now $\mathbb{Z}_p(r-1)_{G_K}$ is a cyclic $\mathfrak{A}_r^+$-module and so $\mathrm{Fitt}_{\mathfrak{A}_r^+}(\mathbb{Z}_p(r-1)_{G_K})$ is equal to $\mathrm{Ann}_{\mathfrak{A}_r^+}(\mathbb{Z}_p(r-1)_{G_K}) = e_r^+\,\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_K})$. It is also clear that the ideal $\mathrm{Fitt}_{\mathfrak{A}_r^+}((H_c^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^+)^\vee)$ is contained inside $\mathrm{Ann}_{\mathfrak{A}_r^+}((H_c^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^+)^\vee) = \mathrm{Ann}_{\mathfrak{A}_r^+}(H_c^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^+)$. To deduce the displayed inclusion in Proposition 4.6.1 from (4.30) it thus suffices to show that any element of $\mathrm{Ann}_{\mathfrak{A}_r^+}(H_c^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^+) = \mathrm{Ann}_{\mathbb{Z}_p[G]}(H_c^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r)))e_r^+$ annihilates $Ш(\mathbb{Z}_p(r)_K)e_r^+$. But this follows immediately from the surjection $H_c^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r)) \twoheadrightarrow \mathrm{Sel}(\mathbb{Z}_p(1-r)_K)^\vee$ of Lemma 3.2.4(i), the natural surjection $\mathrm{Sel}(\mathbb{Z}_p(1-r)_K) \twoheadrightarrow \mathrm{Sel}(\mathbb{Z}_p(1-r)_K)_{\mathrm{cotor}} \cong Ш(\mathbb{Z}_p(1-r)_K)$ and the isomorphism $Ш(\mathbb{Z}_p(1-r)_K)^\vee \cong Ш(\mathbb{Z}_p(r)_K)$ of Proposition 3.2.3.

In an entirely similar way, if $r > 1$ and the Sylow p-subgroup of G is cyclic, then the final assertion of Proposition 4.6.1 follows by combining (4.30) with Lemma 2.1.8 and claims (i) and (ii) of Lemma 3.2.4. $\square$

We are now ready to prove Theorem 4.1.5. To do this we fix an integer r which is strictly greater than one and such that Schneider's conjecture holds for K at r and p. It is at first clear that Theorem 4.1.5(ii) is an immediate consequence of the final assertion of Proposition 4.6.1 and that the displayed inclusion of Proposition 4.6.1 reduces the proof of Theorem 4.1.5(i) to a proof that the ideal $\mathfrak{L}_{p,S}(1-r)\,\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(-r)_{G_K})$ is contained within $\mathrm{Ann}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(r)_K))e_r^-$. But this follows by taking the displayed inclusion in Proposition 4.6.1 with r replaced by $1-r$, noting that $e_{1-r}^+ = e_r^-$ and using the equality $\mathrm{Ann}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(1-r)_K)) = \mathrm{Ann}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(r)_K))$ which is induced by the isomorphism $\text{Ш}(\mathbb{Z}_p(1-r)_K)^\vee \cong \text{Ш}(\mathbb{Z}_p(r)_K)$ in Proposition 3.2.3.

This completes our proof of Theorem 4.1.5.

Chapter 5

The ETNC and Solomon-type Annihilators

Let K/k be a finite abelian extension of a real field k. As in §2.2.1 we set $G := \mathrm{Gal}(K/k)$ and write $S = S_{K/k}$ for the (finite) set of places of k comprising those which ramify in K/k, all archimedean places of k, and all places lying above p.

The basic contents of this chapter is as follows. We begin by describing for each integer r the various "realisations" and "motivic cohomology groups" of the "Tate motive" $\mathbb{Q}(r)_K$. For any integer r strictly greater than one and any field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ we then give an explicit description of the conjecture $\mathrm{ETNC}^{(j)}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ discussed in §1.2.1 (this description is certainly well known to the experts).

We next construct a $\mathbb{Z}_p[G]$-submodule $\mathfrak{S}^{(j)}_{K/k,S,p,r}$ of $\mathbb{C}_p[G]$ from the value at $s = r$ of the S-truncated complex L-function of K/k. This "higher Solomon ideal" is a natural generalisation of the ideal $\mathfrak{S}_{K/k,S,p}$ constructed by Solomon in [55] using the leading term at $s = 1$ of S-truncated complex L-functions. We conjecture that the ideal $\mathfrak{S}^{(j)}_{K/k,S,p,r}$ belongs to $\mathbb{Z}_p[G]$ and annihilates the Tate-Safarevic group $Ш(\mathbb{Z}_p(r)_K)$ and we prove that $\mathrm{ETNC}^{(j)}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ implies the validity of this "higher Solomon conjecture". This result is a natural analogue of the main theorem in the recent thesis of Andrew Jones [34].

5.1 The Motive $\mathbb{Q}(r)_K$

Let r be an integer. We abbreviate $\mathbb{Q}(r)_K$ to M_r and regard this "motive" as defined over k and with a natural action of $\mathbb{Q}[G]$. As in the survey article [26] (where M_r is denoted $h^0(\mathrm{Spec}(K))(r)$) and [13, §2] we shall now describe M_r as a compatible collection of data, referred to either as "realisations" or "motivic cohomology groups". Further details can also be found, for example, in [10, §1.1].

Datum 1 (The Motivic L-function)**.** *Associated to M_r there is an S-truncated motivic L-function*

$$L_S(M_r, s) := \left(L_S(\chi, s+r)\right)_\chi$$

where χ runs over the irreducible complex characters of G.

Datum 2 (The Betti Realisation)**.** *For each embedding $\sigma : k \hookrightarrow \mathbb{R}$ with an associated place v of k, there is associated to M_r a natural $G \times G_v$-module*

$$H_\sigma(M_r) := \bigoplus_{\substack{\delta: K \hookrightarrow \mathbb{C} \\ \delta|_k = \sigma}} (2\pi i)^r \mathbb{Q}$$

upon which G acts by precomposition with the indexing maps, and G_v acts diagonally by post-composition and action on the coefficients. We thereby obtain a $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module by setting

$$H_B(M_r) := \bigoplus_v H_v(M_r)$$

where the direct sum runs over all archimedean places v of k and for each such v the module $H_v(M_r)$ denotes $H_\sigma(M_r)$ for the embedding $\sigma : k \hookrightarrow \mathbb{R}$ that corresponds to v. We denote the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants of $H_B(M_r)$ by $H_B^+(M_r)$.

We note that

$$Y_{K,r} \otimes_{\mathbb{Z}} \mathbb{Q} = H_B(M_r) \tag{5.1}$$

where $Y_{K,r}$ is the $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module defined in §2.2.1. In particular, if K is a CM field, then by combining Lemmas 4.2.2(ii) with the isomorphism (4.29) we deduce that $H_B^+(M_r) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a free $\mathbb{Q}_p[G]e_r^-$-module of rank $[k : \mathbb{Q}]$.

Datum 3 (The deRham Realisation)**.** *There is associated to M_r a finite dimensional filtered K-vector-space*

$$H_{dR}(M_r) := K$$

with a naturally decreasing filtration given by

$$F^i H_{dR}(M_r) := \begin{cases} K & \text{if } i < 1 - r \\ 0 & \text{if } i \geq 1 - r, \end{cases}$$

and an associated "tangent space"

$$t(M_r) := H_{dR}(M_r)/F^0 H_{dR}(M_r) = \begin{cases} K & \text{if } r \geq 1 \\ 0 & \text{if } r < 1. \end{cases} \tag{5.2}$$

Datum 4 (The Motivic Cohomology Spaces)**.** *There are associated to M_r two "motivic cohomology" spaces*

$$H_f^0(k, M_r) := \begin{cases} \mathbb{Q} & \text{if } r = 0 \\ 0 & \text{if } r \neq 0, \end{cases}$$

and

$$H^1_f(k, M_r) := \begin{cases} K_{2r-1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q} & \textit{if } r \geq 1 \\ 0 & \textit{if } r < 1. \end{cases}$$

The canonical isomorphism $K \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{K \hookrightarrow \mathbb{C}} \mathbb{C}$ induces a period isomorphism $H_B(M_r) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{dR}(M_r) \otimes_{\mathbb{Q}} \mathbb{C}$ between the Betti and deRham realisations of M_r (cf. [26, §2, Ex. b]). This isomorphism in turn restricts to give a canonical homomorphism of $\mathbb{R}[G]$-modules

$$\alpha_{M_r} : H^+_B(M_r) \otimes_{\mathbb{Q}} \mathbb{R} \to t(M_r) \otimes_{\mathbb{Q}} \mathbb{R}. \tag{5.3}$$

The "Kummer dual" $M^*(1)$ of M identifies with M_{1-r} and so there is a canonical six-term exact sequence

$$\begin{aligned} 0 \to H^0_f(k, M_r) \otimes_{\mathbb{Q}} \mathbb{R} \to \mathrm{Ker}(\alpha_{M_r}) \to H^1_f(k, M_{1-r})^* \otimes_{\mathbb{Q}} \mathbb{R} \to \\ H^1_f(k, M_r) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{R} \mathrm{Cok}(\alpha_{M_r}) \to H^0_f(k, M_{1-r})^* \otimes_{\mathbb{Q}} \mathbb{R} \to 0 \end{aligned} \tag{5.4}$$

where R is induced by the "Beilinson regulator" map (as described in [9, §10]).

5.2 $\mathrm{ETNC}^{(j)}\left(\mathbb{Q}(r)_K, \mathbb{Z}_p[G]\right)$ for an integer $r > 1$

We now fix an integer r that is strictly greater than one. In this section we give an explicit description of the conjecture $\mathrm{ETNC}^{(j)}(M_r, \mathbb{Z}_p[G])$.

5.2.1 The Archimedean Contribution

In the next result we use the homomorphism α_{M_r} from (5.3).

Lemma 5.2.1.

(i) There is a canonical short exact sequence of $\mathbb{R}[G]$*-modules*

$$0 \to Y_{K,r}^{+} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\alpha_{M_r}} K \otimes_{\mathbb{Q}} \mathbb{R} \to Y_{K,1-r}^{+} \otimes_{\mathbb{Z}} \mathbb{R} \to 0.$$

(ii) The exact sequence (5.4) *specialises to give the canonical isomorphism of* $\mathbb{R}[G]$*-modules*

$$R_{K,r} : K_{2r-1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{R} \cong Y_{K,1-r}^{+} \otimes_{\mathbb{Z}} \mathbb{R}$$

that is induced by the Beilinson regulator map.

Proof. Since $r > 1$ the motivic cohomology spaces $H_f^0(k, M_r)$, $H_f^0(k, M_{1-r})$ and $H_f^1(k, M_{1-r})$ all vanish and the space $H_f^1(k, M_r)$ is equal to $K_{2r-1}(\mathcal{O}_K)\otimes_{\mathbb{Z}}\mathbb{Q}$. The exact sequence (5.4) therefore implies that the homomorphism α_{M_r} is injective and that the homomorphism R is bijective. To prove both claims (i) and (ii) it therefore suffices to prove that there is a canonical isomorphism of $\mathbb{R}[G]$-modules $\mathrm{Cok}(\alpha_{M_r}) \cong Y_{K,1-r}^{+}\otimes_{\mathbb{Z}}\mathbb{R}$.

Now, since $\mathbb{C} = (2\pi i)^{1-r}\mathbb{R} \oplus (2\pi i)^r\mathbb{R}$, there are canonical isomorphisms

$$\begin{aligned}
t(M_r) \otimes_{\mathbb{Q}} \mathbb{R} &= K \otimes_{\mathbb{Q}} \mathbb{R} \\
&\cong (K \otimes_{\mathbb{Q}} \mathbb{C})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \\
&\cong \left(\bigoplus_{K \hookrightarrow \mathbb{C}} \mathbb{C}\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \\
&= \bigoplus_{K \hookrightarrow \mathbb{R}} \mathbb{R} \oplus \left(\bigoplus_{\sigma} \left((2\pi i)^{1-r}\mathbb{R} \oplus (2\pi i)^r\mathbb{R}\right)\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \\
&= \left(\bigoplus_{K \hookrightarrow \mathbb{R}} (2\pi i)^{1-r}\mathbb{R}\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \oplus \left(\bigoplus_{K \hookrightarrow \mathbb{R}} (2\pi i)^{r}\mathbb{R}\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \\
&\qquad \oplus \left(\bigoplus_{\sigma} \left((2\pi i)^{1-r}\mathbb{R} \oplus (2\pi i)^r\mathbb{R}\right)\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \\
&= (Y_{K,r}^{+} \otimes_{\mathbb{Z}} \mathbb{R}) \oplus (Y_{K,1-r}^{+} \otimes_{\mathbb{Z}} \mathbb{R})
\end{aligned}$$

where σ runs over all embeddings $K \hookrightarrow \mathbb{C}$ which do not factor through $\mathbb{R}$ and the last equality follows directly from the explicit descriptions of $Y_{K,1-r}$ and $Y_{K,r}$ in §2.2.1. Since (5.1) implies that the image of α_{M_r} is equal to $Y_{K,r}^{+} \otimes_{\mathbb{Z}} \mathbb{R}$ the above decomposition therefore induces an identification of $\mathrm{Cok}(\alpha_{M_r})$ with $Y_{K,1-r}^{+} \otimes_{\mathbb{Z}} \mathbb{R}$. □

Definition 5.2.2. We define a graded $\mathbb{Q}[G]$-module

$$\Xi(M_r) := [K_{2r-1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q}]_{\mathbb{Q}[G]}^{-1} \otimes_{\mathbb{Q}[G]} \left[Y_{K,r}^{+} \otimes_{\mathbb{Z}} \mathbb{Q}\right]_{\mathbb{Q}[G]}^{-1} \otimes_{\mathbb{Q}[G]} [K]_{\mathbb{Q}[G]}$$

and write $\vartheta_{\infty,r}$ for the composite isomorphism of graded $\mathbb{R}[G]$-modules

$$\begin{aligned}\vartheta_{\infty,r} : \Xi(M_r) \otimes_{\mathbb{Q}} \mathbb{R} \cong & \big[Y^{+}_{K,1-r} \otimes_{\mathbb{Z}} \mathbb{R}\big]^{-1}_{\mathbb{R}[G]} \otimes_{\mathbb{R}[G]} \big[Y^{+}_{K,r} \otimes_{\mathbb{Z}} \mathbb{R}\big]^{-1}_{\mathbb{R}[G]} \\ & \otimes_{\mathbb{R}[G]} [K \otimes_{\mathbb{Q}} \mathbb{R}]_{\mathbb{R}[G]}. \\ \cong & (\mathbb{R}[G], 0)\end{aligned}$$

that is induced by the isomorphism $R_{K,r}$ in Lemma 5.2.1(ii) and the short exact sequence in Lemma 5.2.1(i).

5.2.2 The p-adic Contribution

Proposition 5.2.3. *There exists a canonical isomorphism of graded $\mathbb{Q}_p[G]$-modules*

$$\vartheta_{p,r} : \Xi(M_r) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong [R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))]_{\mathbb{Q}_p[G]}.$$

Proof. Since every bounded complex of $\mathbb{Q}_p[G]$-modules with finitely generated cohomology groups is cohomologically perfect the canonical isomorphism of Lemma 2.1.4 applies with $\mathfrak{A} = \mathbb{Q}_p[G]$ and $C^{\bullet} = R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$. Taking account of the long exact cohomology sequence of the exact triangle (3.1) we therefore obtain a canonical isomorphism of graded $\mathbb{Q}_p[G]$-modules of the form

$$\begin{aligned}&[R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))]_{\mathbb{Q}_p[G]} \cong \\ &\qquad \bigotimes_{i\in\mathbb{Z}} \big[H^i(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))\big]^{(-1)^i}_{\mathbb{Q}_p[G]} \otimes_{\mathbb{Q}_p[G]} \bigotimes_{i\in\mathbb{Z}} \big[P^i(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))\big]^{(-1)^{i+1}}_{\mathbb{Q}_p[G]}. \quad (5.5)\end{aligned}$$

In addition, the definition of $\Xi(M_r)$ immediately gives a canonical isomorphism

$$\Xi(M_r)\otimes_{\mathbb{Q}}\mathbb{Q}_p \cong [K_{2r-1}(\mathcal{O}_K)\otimes_{\mathbb{Z}}\mathbb{Q}_p]^{-1}_{\mathbb{Q}_p[G]}\otimes_{\mathbb{Q}_p[G]}\left[Y^{+}_{K,r}\otimes_{\mathbb{Z}}\mathbb{Q}_p\right]^{-1}_{\mathbb{Q}_p[G]} \otimes_{\mathbb{Q}_p[G]}[K\otimes_{\mathbb{Q}}\mathbb{Q}_p]_{\mathbb{Q}_p[G]}.$$

The claimed isomorphism $\vartheta_{p,r}$ therefore results from combining the last two isomorphisms with the descriptions of cohomology groups given in Lemma 5.2.4 below. $\square$

Lemma 5.2.4.

(i) There are canonical isomorphisms of $\mathbb{Q}_p[G]$-modules

$$H^i(\mathcal{O}_{K,S},\mathbb{Q}_p(r))\cong\begin{cases}K_{2r-1}(\mathcal{O}_K)\otimes_{\mathbb{Z}}\mathbb{Q}_p, & \text{if } i=1\\ 0, & \text{otherwise.}\end{cases}$$

(ii) There are canonical isomorphisms of $\mathbb{Q}_p[G]$-modules

$$P^i(\mathcal{O}_{K,S},\mathbb{Q}_p(r))\cong\begin{cases}Y^{+}_{K,r}\otimes_{\mathbb{Z}}\mathbb{Q}_p, & \text{if } i=0\\ K\otimes_{\mathbb{Q}}\mathbb{Q}_p, & \text{if } i=1,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. We first recall that the p-cohomological dimension of $G_{K,S}$ and of G_w for any place w of K is equal to 2 (cf. [45, Thms. 7.1.8 & 10.9.3]) and hence that the complexes $C^\bullet := R\Gamma(\mathcal{O}_{K,S},\mathbb{Q}_p(r))$ and $R\Gamma(K_w,\mathbb{Q}_p(r))$ for each w in $S(K)$ are acyclic outside of degrees 0, 1 and 2. We also know that the group $H^0(C^\bullet)$ vanishes since $r\neq 0$ and that the group $H^2(C^\bullet)$ vanishes by Lemma 3.2.5.

To prove claim (i) it is therefore enough to note that the claimed isomorphism in degree 1 follows from the "Chern-class" isomorphism $K_{2r-1}(\mathcal{O}_{K,S}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \cong H^1(C^\bullet)$ discussed just prior to Lemma 3.2.5 and the fact that $K_{2r-1}(\mathcal{O}_K) = K_{2r-1}(\mathcal{O}_{K,S})$ since $r > 1$ (cf. [17, Prop. 5.7]).

The isomorphism of claim (ii) in degree 0 was proved in the course of the proof of Proposition 4.4.1(ii). The fact that $P^2(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ vanishes can be seen in the following way: since $K_{2r-2}(\mathcal{O}_{K,S})$ is finite, the Chern-class isomorphism $K_{2r-2}(\mathcal{O}_{K,S}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \cong H^2(C^\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ shows that $H^2(C^\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ vanishes; from Proposition 4.4.1(iv) we know that $H^3_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ vanishes; the exact cohomology sequence of the triangle (3.1) gives an exact sequence $H^2(C^\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to P^2(\mathcal{O}_{K,S}, \mathbb{Q}_p(r)) \to H^3_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ and hence $P^2(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ does indeed vanish.

To complete the proof of claim (ii) it therefore suffices to describe the isomorphism in degree 1. But from Remark 3.1.9 we know that $P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) = P^1_f(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ and so the argument of [6, §3] shows that the Bloch-Kato exponential map $\exp^{\mathrm{BK}}_{p,r} : K \otimes_{\mathbb{Q}} \mathbb{Q}_p \to P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ is bijective. The required isomorphism is then given by the inverse

$$\log^{BK}_{p,r} : P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r)) \to K \otimes_{\mathbb{Q}} \mathbb{Q}_p \tag{5.6}$$

of this "exponential map", which (for obvious reasons) we refer to as the "Bloch-Kato logarithm map". □

5.2.3 Explicit statement of the ETNC

In the following result we use the module $\Xi(M_r)$ and isomorphism $\vartheta_{\infty,r}$ from Definition 5.2.2 and the isomorphism $\vartheta_{p,r}$ from Proposition 5.2.3.

Proposition 5.2.5. *The statement of* $\mathrm{ETNC}^{(j)}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ *is valid if and only if* *one has*

$$\mathbb{Z}_p[G](\vartheta_{p,r} \otimes_{\mathbb{Q}_p} \mathbb{C}_p) \circ (\vartheta_{\infty,r}^{-1} \otimes_{\mathbb{R},j} \mathbb{C}_p) \left(L_S^*(M_r, 0)^{-1}, 0\right) = [R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))]_{\mathbb{Z}_p[G]}.$$

Proof. This is proved by Burns and Flach in [12, pp. 480-481]. □

If the field K is CM, then we shall denote by $\mathrm{ETNC}^{(j),\pm}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ the weaker conjectural equalities

$$\begin{aligned}\mathbb{Z}_p[G]e_r^{\pm}(\vartheta_{p,r} \otimes_{\mathbb{Q}_p} \mathbb{C}_p) \circ (\vartheta_{\infty,r}^{-1} \otimes_{\mathbb{R},j} \mathbb{C}_p) &\left(L_S^*(M_r, 0)^{-1}, 0\right) \\ &= \left[R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))\, e_r^{\pm}\right]_{\mathbb{Z}_p[G]e_r^{\pm}}.\end{aligned}$$

5.3 The Higher Solomon Ideal

We fix a CM abelian extension K of a totally real field k with $G := \mathrm{Gal}(K/k)$ and an integer r that is strictly greater than one. We fix an odd prime p and an isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ and set $d := [k : \mathbb{Q}]$. We write $S = S_{K/k}$ for the set of all places of k which ramify in K/k and all places lying above p.

We recall that in [62, Chap. IV] Tate defines a meromorphic $\mathbb{C}[G]$-valued function $\Theta_{K/k,S}$ with the property that for any complex number s with real part greater than one there is an equality

$$\Theta_{K/k,S}(s) = \prod_{w \notin S(K)} (1 - Nw^{-s} f_w^{-1})^{-1} \in \mathbb{C}[G].$$

In particular, for any character χ of G there is an equality of meromorphic functions

$$\chi(\Theta_{K/k,S}(s)) = L_{K/k,S}(\chi^{-1}, s) \tag{5.7}$$

(cf. [55, Eqn. (5)]). One also knows that the function $L_{K/k,S}(\chi^{-1}, s)$ is analytic on $\mathbb{C}\backslash\{1\}$ and that

$$\mathrm{ord}_{s=1}\left(\chi\left(\Theta_{K/k,S}(s)\right)\right) = \begin{cases} 0 & \text{if } \chi \neq \chi_{0,G} \\ -1 & \text{if } \chi = \chi_{0,G} \end{cases}$$

where $\chi_{0,G}$ denotes the trivial character of G.

We shall now use the value of $\Theta_{K/k,S}(s)$ at $s = r$ to define a $\mathbb{Z}_p[G]$-submodule of $\mathbb{Q}_p[G]$ that is a natural analogue of a construction of Solomon in [55, §2.4]. However, before defining this 'Higher Solomon Ideal' we must first introduce some convenient notation.

- We set
$$a^{-}_{K/k,S,r} := \left(\frac{i}{\pi}\right)^{rd} e^{-}_{r}\Theta_{K/k,S}(r) \in \mathbb{C}\left[G\right]. \tag{5.8}$$

- For any integer i with $1 \leq i \leq d$ we use the embedding $\widetilde{\tau}_i$ introduced in Lemma 4.2.2(ii) and the Bloch-Kato logarithm $\log^{BK}_{p,r}$ from (5.6) to define a homomorphism of $\overline{\mathbb{Q}}_p[G]$-modules

$$\begin{aligned} \lambda^{(j)}_{i,p,r} : P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r)) &\to \overline{\mathbb{Q}}_p[G] \\ a &\mapsto \sum_{g\in G}\left(j\circ\widetilde{\tau}_i\circ g\left(\log^{BK}_{p,r}(a)\right)\right)g^{-1}. \end{aligned}$$

- We define p-adic regulator maps

$$\begin{aligned} R^{(j)}_{K/k,p,r} : {\bigwedge}^{d}_{\mathbb{Z}_p[G]} P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) &\to \overline{\mathbb{Q}}_p[G] \\ a_1\wedge a_2\wedge\cdots\wedge a_d &\mapsto \det\left((\lambda^{(j)}_{i,p,r}(\rho_r(a_t)))_{1\leq i,t\leq d}\right) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{s}^{(j)}_{K/k,S,p,r} : \bigwedge\nolimits^{d}_{\mathbb{Z}_p[G]} P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) &\to \overline{\mathbb{Q}}_p[G] \\ \eta &\mapsto j\left((a^{-}_{K/k,S,r})^{\#}\right) R^{(j)}_{K/k,p,r}(\eta), \end{aligned}$$

where $\rho_r : P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$ is the natural scalar extension and $x \mapsto x^{\#}$ denotes the involution on $\mathbb{C}[G]$ which sends any element $g \in G$ to its inverse.

Definition 5.3.1. In terms of the above notation we now define a 'Higher Solomon Ideal' by setting

$$\mathfrak{S}^{(j)}_{K/k,S,p,r} := \mathrm{Im}(\mathfrak{s}^{(j)}_{K/k,S,p,r}) \subseteq \overline{\mathbb{Q}}_p[G].$$

Remark 5.3.2. Let m be any strictly positive integer. If $m > 1$, then one has $P^1_f(\mathcal{O}_{K,S}, \mathbb{Z}_p(m)) = P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(m))$ (see Remark 3.1.9) but if $m = 1$, then it is shown by Bloch and Kato in [6, Ex. 3.9] that the image of $P^1_f(\mathcal{O}_{K,S}, \mathbb{Z}_p(m))$ in $P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(m))$ identifies naturally with $\prod_{w|p} U^1(K_w)$ where $U^1(K_w)$ denotes the group of principal units of K_w. Furthermore, under the natural topological ring isomorphism $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{w|p} K_w$ the group $\prod_{w|p} U^1(K_w)$ identifies with the Sylow p-subgroup of the group of invertible elements of $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Hence, if we make the same construction as above in the case $r = 1$ and replace the Bloch-Kato logarithm map $\log^{BK}_{p,r}$ by the restriction to $\prod_{w|p} U^1(K_w)$ of the classical p-adic logarithm, then we recover the ideal $\mathfrak{S}^{(j)}_{K/k,S,p}$ defined by Solomon in [55, §2.4]. This shows that the ideal $\mathfrak{S}^{(j)}_{K/k,S,p,r}$ defined above is indeed a natural analogue of Solomon's original construction.

Remark 5.3.3. In [56, Eqn. (22) & Prop 3.4] Solomon proves that his ideal $\mathfrak{S}^{(j)}_{K/k,S,p}$ is both independent of the choice of j and contained in $\mathbb{Q}_p[G]$. For a proof of the same results for the higher Solomon ideal see Appendix A. In particular, we shall often simply write $\mathfrak{S}_{K/k,S,p}$ in place of $\mathfrak{S}^{(j)}_{K/k,S,p}$.

In [55, §3 p.15] Solomon formulates a natural p-adic integrality conjecture for his ideal. A strengthened form of this conjecture was formulated and studied by Andrew Jones in his thesis [34] (see the discussion in §1.1.2) and this work of Jones naturally suggests the following conjecture.

Conjecture 5.3.4 (Higher Solomon Conjecture)**.**

$$\mathfrak{S}_{K/k,S,p,r} \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]}(\text{Ш}(\mathbb{Z}_p(r)_K)).$$

5.4 Statement of the Main Results

We can now state the main results of this chapter. This result is a natural analogue of a theorem of Jones [34, Cor. 4.1.5] and its proof will be given in §5.7.

Theorem 5.4.1. *Let K be a CM abelian extension of a totally real field k and set $G := \mathrm{Gal}(K/k)$. Fix an integer r that is strictly greater than one, an odd prime p and an isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ and assume that $\mathrm{ETNC}^{(j),-}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ is valid.*

(i) One has $\mathfrak{S}^{(j)}_{K/k,S,p,r} \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]}\left(\text{Ш}(\mathbb{Z}_p(r)_K)\right)$.

(ii) If $p \nmid |G|$ then one has

$$\begin{aligned}\mathfrak{S}^{(j)}_{K/k,S,p,r} \cdot \mathrm{Fitt}_{\mathbb{Z}_p[G]}&\left((\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_K}\right)\\ &= \mathrm{Fitt}_{\mathbb{Z}_p[G]}\left(\text{Ш}(\mathbb{Z}_p(r)_K)\right) \cdot \\ &\qquad \mathrm{Fitt}_{\mathbb{Z}_p[G]}\Big(\bigoplus_{\substack{w\in S(K)\\ w\nmid\infty}} (\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_w}\Big)e_r^-.\end{aligned}$$

(iii) If the Quillen-Lichtenbaum conjecture (Conjecture 3.2.6) is valid for r and $n = 2$, then we may replace the Tate-Shafarevic group $Ш(\mathbb{Z}_p(r)_K)$ in claims (i) and (ii) by the p-adic wild kernel $K^w_{2r-2}(\mathcal{O}_K)_p$ of Banaszak.

Corollary 5.4.2. *Let K be an imaginary abelian extension of $\mathbb{Q}$ and k any real subfield of K. Let r be an integer strictly greater than one. Then for each odd prime p one has*

$$\mathfrak{S}_{K/k,S,p,r} \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]}\left(Ш(\mathbb{Z}_p(r)_K)\right).$$

Further, if the Quillen-Lichtenbaum conjecture (Conjecture 3.2.6) is valid for K at r and p (for example, if $r = 2$), then

$$\mathfrak{S}_{K/k,S,p,r} \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]}\left(K^w_{2r-2}(\mathcal{O}_K)_p\right).$$

Proof. Under the stated hypothesis on K it is proved by Burns and Flach in [13, Cor. 1.2] that the statement of $\mathrm{ETNC}^{(j)}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$, and hence also the statement of $\mathrm{ETNC}^{(j),-}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$, is valid. The claimed results therefore follow immediately from Theorem 5.4.1. □

5.5 The Key Algebraic Result

To prove Theorem 5.4.1 we shall combine the (conjectural) equality of Proposition 5.2.5 with an explication of the map $(\vartheta_{p,r} \otimes_{\mathbb{Q}_p} \mathbb{C}_p) \circ (\vartheta^{-1}_{\infty,r} \otimes_{\mathbb{R},j} \mathbb{C}_p)$ which occurs in that equality, the explicit description of cohomology given in Proposition 4.4.4 and a purely algebraic result of Jones.

In this section we start the proof by giving a convenient reformulation of the algebraic result of Jones.

Proposition 5.5.1. *Let G be a finite abelian group, let $\mathfrak{A}$ be a direct factor of $\mathbb{Z}_p[G]$ and set $A := \mathfrak{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $C^\bullet$ be a complex which satisfies the following four conditions.*

(i) $C^\bullet$ is a perfect complex of $\mathfrak{A}$-modules.

(ii) The Euler characteristic of $C^\bullet \otimes_{\mathfrak{A}} A$ in $K_0(A)$ is zero.

(iii) $C^\bullet$ is acyclic outside of degrees 1 and 2.

(iv) $H^1(C^\bullet)$ is a free $\mathfrak{A}$-module of rank d (for some non-negative integer d).

Let M be any $\mathfrak{A}$-module for which there exists an $\mathfrak{A}$-module homomorphism

$$\theta : M \to H^2(C^\bullet)$$

with both finite kernel and finite cokernel and write ρ for the natural scalar extension $M \to M \otimes_{\mathfrak{A}} A$. Then there is a natural injection of graded $\mathfrak{A}$-modules

$$\left[H^1(C^\bullet)\right]_{\mathfrak{A}}^{-1} \cdot \left(\bigwedge\nolimits_{\mathfrak{A}}^{d} \mathrm{Im}((\theta \otimes \mathbb{Q}_p) \circ \rho), 0\right) \hookrightarrow (\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Cok}(\theta)), 0) \cdot [C^\bullet]_{\mathfrak{A}}.$$

Furthermore, if $\mathfrak{A}$ is the maximal $\mathbb{Z}_p$-order in A, then there is a natural isomorphism of graded $\mathfrak{A}$-modules

$$\left[H^1(C^\bullet)\right]_{\mathfrak{A}}^{-1} \cdot \left(\bigwedge\nolimits_{\mathfrak{A}}^{d} \mathrm{Im}((\theta \otimes \mathbb{Q}_p) \circ \rho), 0\right) \cong (\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Im}(\theta)_{\mathrm{tor}}) \cdot \mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Cok}(\theta)), 0) \cdot [C^\bullet]_{\mathfrak{A}}.$$

Proof. By Lemma 5.5.2 (below) we know that the $\mathfrak{A}$-module $H^2(C^\bullet)$ has finite projective dimension. Also, since $C^\bullet \otimes^{\mathbb{L}}_{\mathfrak{A}} A$ is acyclic outside degrees 1 and 2 and has zero Euler characteristic in $K_0(A)$ the A-module $H^2(C^\bullet) \otimes_{\mathfrak{A}} A$ is isomorphic to $H^1(C^\bullet) \otimes_{\mathfrak{A}} A$ and so is free of rank d. We may therefore apply Lemma 5.5.3 (below) with $L = \mathrm{Im}(\theta), M = H^2(C^\bullet)$ and $C = \mathrm{Cok}(\theta)$ to deduce that

$$\mathrm{Im}(\kappa) \subseteq (\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Cok}(\theta)), 0) \cdot \left[H^2(C)\right]_{\mathfrak{A}} \tag{5.9}$$

where κ is the natural inclusion of $\left(\left(\bigwedge_{\mathfrak{A}}^d \mathrm{Im}(\theta)\right)_{\mathrm{tf}}, d\right)$ in $[H^2(C) \otimes_{\mathfrak{A}} A]_A$. Furthermore if $\mathfrak{A}$ is the maximal $\mathbb{Z}_p$-order in A, then the same lemma gives an equality

$$\mathrm{Im}(\kappa) = (\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Im}(\theta)_{\mathrm{tor}}) \cdot \mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Cok}(\theta)), 0) \cdot \left[H^2(C)\right]_{\mathfrak{A}}. \tag{5.10}$$

Now since $\theta(M_{\mathrm{tor}}) \subseteq (\mathrm{Im}(\theta))_{\mathrm{tor}}$ there is a commutative diagram

$$\begin{tikzcd}
M \otimes_{\mathfrak{A}} A \arrow[r, "\theta\otimes A", "\cong"'] & H^2(C) \otimes_{\mathfrak{A}} A \\
M \arrow[u, "\rho"] \arrow[r, two heads] & \mathrm{Im}(\theta)_{\mathrm{tf}} \arrow[u, hook]
\end{tikzcd}$$

where the bottom map is induced by θ and the right-hand map is the natural injection which induces κ. Hence one has $\mathrm{Im}(\kappa) = (\bigwedge_{\mathfrak{A}}^d \mathrm{Im}((\theta \otimes \mathbb{Q}_p) \circ \rho), d)$.

Next we note that since $H^1(C)$ and $H^2(C)$ are of finite projective dimension the complex $C^\bullet$ is cohomologically perfect and so Lemma 2.1.4 gives a canonical isomorphism of graded $\mathfrak{A}$-modules of the form

$$\left[H^2(C^\bullet)\right]_{\mathfrak{A}} \cong [C^\bullet]_{\mathfrak{A}} \otimes_{\mathfrak{A}} \left[H^1(C^\bullet)\right]_{\mathfrak{A}}.$$

To obtain the desired inclusion, resp. equality, we now simply substitute this isomorphism into (5.9), resp. (5.10). □

Lemma 5.5.2. *Let $\mathfrak{A}$ be any commutative Noetherian ring and let $C^\bullet$ be a perfect complex of $\mathfrak{A}$-modules that is acyclic outside of degrees 1 and 2. If $H^1(C^\bullet)$ is a* *projective $\mathfrak{A}$-module, then $H^2(C^\bullet)$ is of finite projective dimension over $\mathfrak{A}$.*

Proof. Since $C^\bullet$ is both perfect and acyclic outside of degrees 1 and 2 a standard construction in cohomological algebra shows that there exists an exact sequence of finitely generated $\mathfrak{A}$-modules of the form

$$0 \longrightarrow H^1(C) \xrightarrow{f'} P^1 \xrightarrow{f} P^2 \xrightarrow{f''} H^2(C) \longrightarrow 0$$

in which P^2 is free and P^1 is of finite projective dimension. We choose a finite projective resolution of P^1

$$0 \longrightarrow Q_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\gamma} P^1 \longrightarrow 0.$$

Then, since $H^1(C)$ is projective, there exists a monomorphism $g : H^1(C) \hookrightarrow Q_0$ such that $\gamma \circ g = f'$. We note that since $\gamma \circ g$ is injective the intersection of the image of g with the image of ∂_1 is trivial, and hence the following exact sequence gives a finite projective resolution of $H^2(C)$:

$$0 \longrightarrow Q_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} Q_2 \xrightarrow{(\partial_2,0)} Q_1 \oplus H^1(C) \xrightarrow{\partial_1+g} Q_0 \xrightarrow{f\circ\gamma} P^2 \xrightarrow{f''} H^2(C) \longrightarrow 0.$$

$\square$

Lemma 5.5.3. (A. Jones, [34, Prop. 2.2.2, Rem. 2.2.3])

Let R be a complete discrete valuation ring with field of fractions F, G a finite abelian group and $\mathfrak{A}$ a ring direct summand of $R[G]$. Set $A := \mathfrak{A} \otimes_R F$. Let C be a finite $\mathfrak{A}$-module sitting in a given short exact sequence of $\mathfrak{A}$-modules

$$0 \to L \to M \to C \to 0$$

in which M has finite projective dimension and $M \otimes_{\mathfrak{A}} A$ is a free A-module of rank n. Then there is a natural composite morphism of graded modules

$$\kappa : \left(\left(\bigwedge\nolimits_{\mathfrak{A}}^{n} L\right)_{\mathrm{tf}}, n\right) \hookrightarrow \left(\bigwedge\nolimits_{A}^{n} (L \otimes_{\mathfrak{A}} A), n\right) \cong [M \otimes_{\mathfrak{A}} A]_A$$

and one has

$$\mathrm{Im}(\kappa) \subseteq (\mathrm{Fitt}_{\mathfrak{A}}(C), 0) \cdot [M]_{\mathfrak{A}}.$$

Furthermore, if $\mathfrak{A}$ is the maximal R-order in A, then one has

$$\mathrm{Im}(\kappa) = (\mathrm{Fitt}_{\mathfrak{A}}(L_{\mathrm{tor}}) \cdot \mathrm{Fitt}_{\mathfrak{A}}(C), 0) \cdot [M]_{\mathfrak{A}}.$$

5.6 Explicating the morphism $(\vartheta_{p,r} \otimes_{\mathbb{Q}_p} \mathbb{C}_p) \circ (\vartheta_{\infty,r}^{-1} \otimes_{\mathbb{R},j} \mathbb{C}_p)e_r^-$

In this section we relate the map $(\vartheta_{p,r} \otimes_{\mathbb{Q}_p} \mathbb{C}_p) \circ (\vartheta_{\infty,r}^{-1} \otimes_{\mathbb{R},j} \mathbb{C}_p)e_r^-$ which occurs in the statement of Proposition 5.2.5 to the p-adic regulator maps $R_{K/k,p,r}^{(j)}$ that are used in our definition of the Higher Solomon Ideal. We therefore fix notation as in §5.3. For convenience we also abuse notation throughout this section by often suppressing explicit reference to the fixed isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ and abbreviating $\vartheta_{p,r} \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ and $\vartheta_{\infty,r} \otimes_{\mathbb{R},j} \mathbb{C}_p$ to $\vartheta_{p,r}$ and $\vartheta_{\infty,r}$ respectively.

Before stating our main result we recall from Lemma 3.2.12(iii) that the module $K_{2r-1}(\mathcal{O}_K)e_r^-$ is finite. We may therefore regard the isomorphisms that occur in Lemma 5.2.4 and the proof of Proposition 5.2.3 as inducing an identification

$$[R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))]_{\mathbb{Q}_p[G]} = \left[Y_{K,r}^{+} \otimes_{\mathbb{Z}} \mathbb{Q}_p\right]_{\mathbb{Q}_p[G]}^{-1} \otimes_{\mathbb{Q}_p[G]} \left[P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))\right]_{\mathbb{Q}_p[G]}. \tag{5.11}$$

We also set $\mathfrak{A} := \mathbb{Z}_p[G]e_r^-$ and fix an $\mathfrak{A}$-basis α^{-1} of the ungraded part of $\left[Y_{K,r}^{+} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right]_{\mathfrak{A}}^{-1}$

in the following way. Recall first from (4.8) that, regarding $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$, each embedding $K \hookrightarrow \mathbb{C}$ corresponds to a pair (τ_i, g) where $\{\tau_i : 1 \le i \le d\}$ is a chosen set of embeddings $K \hookrightarrow \mathbb{C}$ and g is an element of G. We let $\{\alpha_i : 1 \le i \le d\}$ be the $\mathbb{Z}[G]$-basis of $Y_{K,r} = \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} (2\pi i)^r \mathbb{Z}$ given by

$$(\alpha_i)_\sigma = \begin{cases} (2\pi i)^r & \text{if } \sigma = \tau_i, \\ 0 & \text{otherwise.} \end{cases}$$

We write α_i^{-1} for the unique element of $\mathrm{Hom}_{\mathbb{Z}[G]}(Y_{K,r}, \mathbb{Z}[G])$ for which $\alpha_i^{-1}(\alpha_j)$ is equal to 1 if $i = j$ and is equal to 0 otherwise. We then obtain an $\mathfrak{A}$-basis of the ungraded part of $\left[Y_{K,r}^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p\right]_{\mathfrak{A}}^{-1}$ by setting

$$\alpha^{-1} := e_r^-(\alpha_1^{-1}) \wedge_{\mathfrak{A}} e_r^-(\alpha_2^{-1}) \wedge_{\mathfrak{A}} \cdots \wedge_{\mathfrak{A}} e_r^-(\alpha_d^{-1}).$$

Proposition 5.6.1. *The composite morphism $\vartheta_{\infty,r} \circ \vartheta_{p,r}^{-1} e_r^-$ is the unique isomorphism of graded modules $\left[R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))e_r^- \otimes_{\mathbb{Q}_p} \mathbb{C}_p\right]_{\mathbb{C}_p[G]e_r^-} \to (\mathbb{C}_p[G]e_r^-, 0)$ with the following property: for each x in the ungraded part of $[P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))e_r^-]_{\mathbb{Q}_p[G]e_r^-}$ one has*

$$\vartheta_{\infty,r} \circ \vartheta_{p,r}^{-1}((\alpha^{-1} \otimes x), 0) = ((2\pi i)^{-rd} R_{K/k,p,r}(x), 0) \in (\mathbb{C}_p[G]e_r^-, 0),$$

where we have used the identification (5.11).

Proof. We set $A = \mathfrak{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p[G]e_r^-$. We note first that specifying $\vartheta_{\infty,r} \circ \vartheta_{p,r}^{-1}$ on elements of the form $((\alpha^{-1} \otimes x), 0)$ is enough since α^{-1} is an A-basis of $\left[Y_{K,r}^+ \otimes_{\mathbb{Z}} \mathbb{Q}_p\right]_A^{-1}$. Since the ungraded part of $[P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))e_r^-]_A$ is equal to $\bigwedge_A^d P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))e_r^-$ it is also clearly enough to consider elements of the form $x = u_1 \wedge \cdots \wedge u_d$ with $u_i \in P^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))e_r^-$ for each i. But by unwinding the definitions of $\vartheta_{\infty,r}$ and $\vartheta_{p,r}$

from Definition 5.2.2 and Proposition 5.2.3 one finds that

$$\vartheta_{\infty,r} \circ \vartheta_{p,r}^{-1}((e_r^-(\alpha_1^{-1}) \wedge \cdots \wedge e_r^-(\alpha_d^{-1})) \otimes (u_1 \wedge \cdots \wedge u_d), 0) \\ = \det((\alpha_i^{-1}(\Pi(\log_{p,r}^{BK}(u_j))))_{1 \le i,j \le d}) \quad (5.12)$$

with Π the natural isomorphism

$$K \otimes_{\mathbb{Q}} \mathbb{C}_p = (K \otimes_{\mathbb{Q}} \mathbb{C}) \otimes_{\mathbb{C}} \otimes \mathbb{C}_p \cong (\prod_{\sigma : K \hookrightarrow \mathbb{C}} \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}_p = \prod_{\sigma : K \hookrightarrow \mathbb{C}} \mathbb{C}_p = Y_{K,r} \otimes_{\mathbb{Z}} \mathbb{C}_p$$

where σ runs over all embeddings $K \hookrightarrow \mathbb{C}$. Now, if we identify each composite $j \circ \sigma$ with the induced embedding $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbb{C}_p$ and regard $\mathrm{Emb}(K, \mathbb{C})$ as a basis of $Y_{K,r} \otimes_{\mathbb{Z}} \mathbb{C}_p$, then for each index b one has

$$\begin{aligned} \Pi(\log_{p,r}^{BK}(u_b)) &= \sum_{\sigma} j \circ \sigma(\log_{p,r}^{BK}(u_b)) \cdot \sigma \\ &= \sum_{a=1}^{d} \sum_{g \in G} (j \circ \tau_a \circ g)(\log_{p,r}^{BK}(u_b)) \cdot g^{-1}(\tau_a) \\ &= (2\pi i)^{-r} \sum_{a=1}^{d} \sum_{g \in G} (j \circ \tau_a \circ g)(\log_{p,r}^{BK}(u_b)) \cdot g^{-1}(\alpha_a). \end{aligned}$$

For each index c one therefore has

$$\begin{aligned} \alpha_c^{-1}(\Pi(\log_{p,r}^{BK}(u_b))) &= (2\pi i)^{-r} \sum_{a=1}^{a=d} (\sum_{g \in G} (j \circ \tau_a \circ g)(\log_{p,r}^{BK}(u_b)) g^{-1}) \alpha_c^{-1}(\alpha_a) \\ &= (2\pi i)^{-r} \sum_{g \in G} (j \circ \tau_c \circ g)(\log_{p,r}^{BK}(u_b)) g^{-1} \\ &= (2\pi i)^{-r} \lambda_{c,p,r}^{j}(u_b) \end{aligned}$$

where the last equality is valid by definition of the map $\lambda_{c,p,r}^{j}$. From (5.12) it therefore follows that

$$
\begin{aligned}
&\vartheta_{\infty,r} \circ \vartheta_{p,r}^{-1}((e_r^-(\alpha_1^{-1}) \wedge \cdots \wedge e_r^-(\alpha_d^{-1})) \otimes (u_1 \wedge \cdots \wedge u_d), 0) \\
=&\det(((2\pi i)^{-r} \lambda_{c,p,r}^j(u_b))_{1\le b,c\le d})) \\
=&(2\pi i)^{-rd} \det((\lambda_{c,p,r}^j(u_b))_{1\le b,c\le d}) \\
=&(2\pi i)^{-rd} R_{K/k,p,r}(u_1 \wedge \cdots \wedge u_d),
\end{aligned}
$$

where the last equality is by the definition of the regulator map $R_{K/k,p,r}$. This is the required equality. □

5.7 The Proof of Theorem 5.4.1

Claim (iii) of Theorem 5.4.1 is an immediate consequence of Lemma 3.2.7 and so we focus on claims (i) and (ii). We recall from Lemma 3.2.4 that there is an isomorphism of $\mathbb{Z}_p[G]$-modules $Ш(\mathbb{Z}_p(r)_K) \cong Ш^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$. Further, from the definition (5.8) of the element $a^-_{K/k,S,r}$ it is clear that $a^-_{K/k,S,r} = a^-_{K/k,S,r} e_r^-$ and hence also that $\mathfrak{S}_{K/k,S,p,r} = \mathfrak{S}_{K/k,S,p,r} e_r^-$. To prove claims (i) and (ii) of Theorem 5.4.1 it is therefore enough to prove that the validity of $\mathrm{ETNC}^{(j),-}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ implies an inclusion

$$
\mathfrak{S}_{K/k,S,p,r} e_r^- \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]}\left(Ш^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))\right) \tag{5.13}
$$

and if $p \nmid |G|$ also an equality

$$
\begin{aligned}
&\mathfrak{S}_{K/k,S,p,r} \,\mathrm{Fitt}_{\mathbb{Z}_p[G]}\left((\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_K}\right) e_r^- \\
&\quad = \mathrm{Fitt}_{\mathbb{Z}_p[G]}\left(Ш^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))\right) \cdot \mathrm{Fitt}_{\mathbb{Z}_p[G]}\Big(\bigoplus_{\substack{w \in S(K) \\ w \nmid \infty}} (\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_w}\Big) e_r^-.
\end{aligned} \tag{5.14}
$$

The key to proving this is the following result.

Lemma 5.7.1. *Proposition 5.5.1 applies to the following data:*

- $C^\bullet := R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$.
- $\mathfrak{A} = \mathfrak{A}^- := \mathbb{Z}_p[G]e_r^-$ *and* $A = A^- := \mathbb{Q}_p[G]e_r^-$.
- $M := P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$ *and* θ *is the homomorphism*

$$M = P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^- \to H_c^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^- = H^2(C^\bullet)$$

  that occurs in the long exact sequence of cohomology of the distinguished triangle (3.1).

With respect to this data one also knows that

- *the module* $H^1(C^\bullet)$ *is canonically isomorphic to* $Y_{K,r}^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p$.
- $d = [k : \mathbb{Q}]$.
- $\mathrm{Cok}(\theta)$ *is isomorphic to the finite module* $\text{Ш}^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$.

Proof. Proposition 4.4.4 implies that this data satisfies the conditions (i)-(iv) of Proposition 5.5.1 and also shows that $H^1(C^\bullet)$ is canonically isomorphic to the free $\mathfrak{A}$-module $Y_{K,r}^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of rank $d := [k : \mathbb{Q}]$.

The long exact sequence of cohomology of the distinguished triangle (3.1) combines with the definition of $\text{Ш}^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ to imply that $\mathrm{Cok}(\theta)$ is equal to the finite module $\text{Ш}^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$. In the same way we find that $\mathrm{Ker}(\theta)$ is isomorphic to a quotient of $H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))e_r^-$ and the latter module is finite because the Chern class map $\mathrm{ch}_{K,S,p,1}^r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ discussed in §3.2.1 is bijective and the module $K_{2r-1}(\mathcal{O}_{K,S})e_r^-$ is finite (see Lemma 3.2.12(iii)). The homomorphism θ therefore has finite kernel and finite cokernel, as required. $\square$

In the remainder of this section we use the notation of Lemma 5.7.1.

Now the regulator map $R_{K/k,S,p,r}$ factors through $\wedge^d_{\mathfrak{A}}\rho$ with ρ the natural scalar extension $P^1(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^- \to P^1(\mathcal{O}_{K,S},\mathbb{Q}_p(r))e_r^-$ and so the formula of Proposition 5.6.1 implies that

$$(\left(\frac{i}{\pi}\right)^{rd}\mathrm{Im}(R_{K/k,S,p,r}),0) = \vartheta_{\infty,r}\circ\vartheta_{p,r}^{-1}(\left[H^1(C^\bullet)\right]_{\mathfrak{A}}^{-1}\otimes_{\mathfrak{A}}\bigwedge\nolimits_{\mathfrak{A}}^{d}\mathrm{Im}((\theta\otimes\mathbb{Q}_p)\circ\rho),d). \quad (5.15)$$

Here we have used the fact that $\frac{i}{\pi} = -2(2\pi i)^{-1}$ and that $-2e_r^-$ is a unit of $\mathfrak{A}$. (Note also that the use of the map $\theta\otimes\mathbb{Q}_p$ is implicit in the formula of Proposition 5.6.1 since we regard (5.11) as an identification.)

In addition, by applying Proposition 5.5.1 in the context of Lemma 5.7.1, and then using the isomorphism $\mathrm{Cok}(\theta)\cong Ш^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^-$ in Lemma 5.7.1 we obtain a natural inclusion

$$\left[H^1(C^\bullet)\right]_{\mathfrak{A}}^{-1}\otimes_{\mathfrak{A}}(\bigwedge\nolimits_{\mathfrak{A}}^{d}\mathrm{Im}((\theta\otimes\mathbb{Q}_p)\circ\rho),0)\hookrightarrow(\mathrm{Fitt}_{\mathfrak{A}}(Ш^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^-,0)\cdot[C^\bullet]_{\mathfrak{A}}$$

and if $p\nmid|G|$, so that $\mathfrak{A}$ is the maximal $\mathbb{Z}_p$-order in A, also a natural isomorphism

$$\begin{aligned}&\left[H^1(C^\bullet)\right]_{\mathfrak{A}}^{-1}\otimes_{\mathfrak{A}}(\bigwedge\nolimits_{\mathfrak{A}}^{d}\mathrm{Im}((\theta\otimes\mathbb{Q}_p)\circ\rho),0)\\ &\qquad\cong(\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Im}(\theta)_{\mathrm{tor}})\,\mathrm{Fitt}_{\mathfrak{A}}(Ш^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^-),0)\cdot[C^\bullet]_{\mathfrak{A}}.\end{aligned}$$

Upon substituting (5.15) into the last two displayed expressions we obtain an inclusion

$$(\left(\frac{i}{\pi}\right)^{rd}\mathrm{Im}(R_{K/k,S,p,r}),0)\subseteq(\mathrm{Fitt}_{\mathfrak{A}}(Ш^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^-),0)\cdot\vartheta_{\infty,r}\circ\vartheta_{p,r}^{-1}([C^\bullet]_{\mathfrak{A}}) \quad (5.16)$$

and if $p \nmid |G|$ an equality

$$\begin{aligned}(\left(\frac{i}{\pi}\right)^{rd} &\mathrm{Im}(R_{K/k,S,p,r}),0)\\ &= (\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Im}(\theta)_{\mathrm{tor}})\,\mathrm{Fitt}_{\mathfrak{A}}(\text{Ш}^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^-),0)\cdot\vartheta_{\infty,r}\circ\vartheta_{p,r}^{-1}([C^\bullet]_{\mathfrak{A}}).\end{aligned} \tag{5.17}$$

Now if $\mathrm{ETNC}^{(j),-}(\mathbb{Q}(r)_K,\mathbb{Z}_p[G])$ is valid, then Proposition 5.2.5 implies that

$$\vartheta_{\infty,r}\circ\vartheta_{p,r}^{-1}([C^\bullet]_{\mathfrak{A}}) = \left(L_S^*\left(M_r,0\right)^{-1}\cdot\mathfrak{A},0\right).$$

But, since $r>1$, the identity (5.7) implies that $L_S^*\left(M_r,0\right)=\Theta_{K/k,S}(r)^{\#}$. Upon substituting the last formula into (5.16), resp. (5.17), we thus obtain an inclusion

$$(\left(\frac{i}{\pi}\right)^{rd} e_r^-\Theta_{K/k,S}(r)^{\#}\,\mathrm{Im}(R_{K/k,S,p,r}),0)\subseteq(\mathrm{Fitt}_{\mathfrak{A}}(\text{Ш}^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^-),0), \tag{5.18}$$

resp. an equality

$$\begin{aligned}(\left(\frac{i}{\pi}\right)^{rd} &e_r^-\Theta_{K/k,S}(r)^{\#}\,\mathrm{Im}(R_{K/k,S,p,r}),0)\\ &= (\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Im}(\theta)_{\mathrm{tor}})\,\mathrm{Fitt}_{\mathfrak{A}}(\text{Ш}^2(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^-),0).\end{aligned} \tag{5.19}$$

But $\left(\frac{i}{\pi}\right)^{rd} e_r^-\Theta_{K/k,S}(r)^{\#}\,\mathrm{Im}(R_{K/k,S,p,r})$ is equal to $\mathfrak{S}_{K/k,S,p,r}$ (by the very definition of the latter ideal in Definition 5.3.1). Taking this into account it is clear that (5.18) immediately gives the required inclusion (5.13) and that (5.19) implies (5.14) when combined with the following result.

Lemma 5.7.2. *One has*

$$\begin{aligned}&\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Im}(\theta)_{\mathrm{tor}})\\ &\qquad = (\mathrm{Fitt}_{\mathbb{Z}_p[G]}((\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_K})e_r^-)^{-1}\,\mathrm{Fitt}_{\mathbb{Z}_p[G]}\Bigg(\bigoplus_{\substack{w\in S(K)\\ w\nmid\infty}}(\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_w}\Bigg)e_r^-.\end{aligned}$$

Proof. The long exact cohomology sequence associated to the distinguished triangle (3.1) gives an exact sequence of $\mathfrak{A}$-modules

$$0\to H^1\left(\mathcal{O}_{K,S},\mathbb{Z}_p(r)\right)e_r^- \xrightarrow{\gamma} P^1\left(\mathcal{O}_{K,S},\mathbb{Z}_p(r)\right)e_r^- \xrightarrow{\theta} \mathrm{Im}(\theta)\to 0. \tag{5.20}$$

(The only thing that is not obvious here is the injectivity of γ and this is true because $\ker(\gamma) = Ш^1(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^-$ vanishes by Lemma 3.2.10(iii).)

Now Lemma 5.7.1 implies that the map $\theta\otimes\mathbb{Q}_p$ is bijective. Since the functor $-_{\mathrm{tor}}$ is left-exact there is thus a commutative diagram of the form

$$\begin{tikzcd}
& & P^1\left(\mathcal{O}_{K,S},\mathbb{Q}_p(r)\right)e_r^- \arrow[r, "\theta\otimes\mathbb{Q}_p", "\cong"'] & H^2_c(\mathcal{O}_{K,S},\mathbb{Q}_p(r))e_r^- & \\
0 \arrow[r] & H^1\left(\mathcal{O}_{K,S},\mathbb{Z}_p(r)\right)e_r^- \arrow[r] & P^1\left(\mathcal{O}_{K,S},\mathbb{Z}_p(r)\right)e_r^- \arrow[r] \arrow[u] & \mathrm{Im}(\theta) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & (H^1\left(\mathcal{O}_{K,S},\mathbb{Z}_p(r)\right)e_r^-)_{\mathrm{tor}} \arrow[r] \arrow[u, "\cong"'] & (P^1\left(\mathcal{O}_{K,S},\mathbb{Z}_p(r)\right)e_r^-)_{\mathrm{tor}} \arrow[r, "g"] \arrow[u] & \mathrm{Im}(\theta)_{\mathrm{tor}} \arrow[u] & \\
& & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

which has exact rows and columns. An easy diagram chase shows that the homomorphism g is surjective and so one has an exact sequence

$$0\to(H^1\left(\mathcal{O}_{K,S},\mathbb{Z}_p(r)\right)e_r^-)_{\mathrm{tor}}\to(P^1\left(\mathcal{O}_{K,S},\mathbb{Z}_p(r)\right)e_r^-)_{\mathrm{tor}}\to\mathrm{Im}(\theta)_{\mathrm{tor}}\to 0.$$

Since Fitting ideals are multiplicative on exact sequences (as $p \nmid |G|$) it follows that

$$\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Im}(\theta)_{\mathrm{tor}}) = \mathrm{Fitt}_{\mathfrak{A}}((P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))\, e_r^-)_{\mathrm{tor}})\, \mathrm{Fitt}_{\mathfrak{A}}((H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))\, e_r^-)_{\mathrm{tor}})^{-1}.$$

To obtain the required description of $\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{Im}(\theta)_{\mathrm{tor}})$ it is thus enough to prove that there are isomorphisms of $\mathbb{Z}_p[G]$-modules of the form $H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))_{\mathrm{tor}} \cong (\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_{K,S}}$ and $P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))_{\mathrm{tor}} \cong \bigoplus_{\substack{w \in S(K) \\ w \nmid \infty}} (\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_w}$.

Now $r \neq 0$ and so $\mathbb{Q}_p(r)^{G_{K,S}}$ vanishes. The tautological short exact sequence

$$0 \to \mathbb{Z}_p(r) \to \mathbb{Q}_p(r) \to \mathbb{Q}_p/\mathbb{Z}_p(r) \to 0 \tag{5.21}$$

thus induces an exact sequence of cohomology groups

$$0 \to (\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_{K,S}} \to H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)) \to H^1(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))$$

and hence also the required isomorphism $(\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_{K,S}} \cong H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))_{\mathrm{tor}}$. Next we note that $H^1(K_w, \mathbb{Z}_p(r)) = 0$ for each archimedean place w since p is odd and hence that $P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))_{\mathrm{tor}} = \bigoplus_{\substack{w \in S(K) \\ w \nmid \infty}} H^1(K_w, \mathbb{Z}_p(r))_{\mathrm{tor}}$. Also for each non-archimedean place w one has $\mathbb{Q}_p(r)^{G_w} = 0$ and so the exact sequence (5.21) induces the required isomorphism $(\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_w} \cong H^1(K_w, \mathbb{Z}_p(r))_{\mathrm{tor}}$. $\square$

Appendices

Appendix A

The independence of $\mathfrak{s}_{K/k,S,p}$ from j

In [34] Jones showed that if K is a CM abelian extension of a totally real field k, then for any isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ the conjecture $\mathrm{ETNC}^{(j),-}(\mathbb{Q}(1)_K, \mathbb{Z}_p[G])$ implies that the Solomon ideal $\mathfrak{S}^{(j)}_{K/k,S,p}$ (cf. Remark 5.3.2) is contained in $\mathbb{Q}_p[G]$. That the Solomon ideal has this property and furthermore is also independent of the choice of j had in fact previously been proved unconditionally by Solomon [56].

From Theorem 5.4.1 we know that if K is a CM abelian extension of a totally real field k, then for any isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ and any integer $r > 1$ the conjecture $\mathrm{ETNC}^{(j),-}(\mathbb{Q}(r)_K, \mathbb{Z}_p[G])$ implies that the Higher Solomon Ideal $\mathfrak{S}^{(j)}_{K/k,S,p,r}$ is contained in $\mathbb{Q}_p[G]$. In this section we adapt Solomon's original techniques to prove *unconditionally* both this inclusion and the fact that $\mathfrak{S}^{(j)}_{K/k,S,p,r}$ is independent of the choice of j.

By abuse of notation we shall also write j for the inclusion $j : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ obtained by restricting j.

A.1 The function $\Phi_{K/k}$

For a more complete account of the basic definitions and theory of Solomon's function $\Phi_{K/k}$ the reader is referred to [54, §2] and [34, §3.2].

For any cycle $\mathfrak{m} = \mathfrak{f}\mathfrak{z}$ in k, let I be the fractional ideal with additive character ξ who's $\mathcal{O}_k$-annihilator, $\mathrm{Ann}_{\mathcal{O}_k}(\xi) = \mathfrak{f}$. In [54, §2.2] Solomon defines a finite set $\mathfrak{W}_\mathfrak{m}$ which consists of the $\mathfrak{z}$ equivalence classes of pairs (ξ, I). This set is endowed with a distinguished element $\mathfrak{w}^0_\mathfrak{m}$ and a free transitive action of the ray class-group $\mathrm{Cl}_\mathfrak{m}(\mathcal{O}_k)$ corresponding to $\mathfrak{m}$.

In [56, §2, Eqn. (7)] Solomon defines the function

$$\Phi_\mathfrak{m}(s) := \sum_{\mathfrak{c} \in \mathrm{Cl}_\mathfrak{m}(k)} Z(s; \mathfrak{c}\mathfrak{w}^0_\mathfrak{m})\sigma_\mathfrak{c}^{-1}$$

where $\sigma_\mathfrak{c}$ is an element of G corresponding to $\mathfrak{c}$ via the Artin map, and for $\mathfrak{w} \in \mathfrak{W}_\mathfrak{m}$, $Z(s; \mathfrak{w})$ denotes the "twisted zeta function", $Z_\emptyset(s; \mathfrak{w})$ defined in [54, Def. 2.1].

It is know that $\Phi_\mathfrak{m}$ is a meromorphic $\mathbb{C}[\mathrm{Gal}(k(\mathfrak{m})/k)]$-valued function with at worst a simple pole at $s = 0$. Hence we may define the $\mathbb{C}[G]$-valued function

$$\Phi_{K/k}(s) := (|d_k| N\mathfrak{f}(K))^{s-1} \pi_{k(\mathfrak{m}(K)),K} \left(\Phi_{\mathfrak{m}(K)}(s)\right) \tag{A.1}$$

where d_k is the absolute discriminant of k and $\pi_{k(\mathfrak{m}(K)),K}$ is the restriction map $\pi_{k(\mathfrak{m}(K)),K} : \mathrm{Gal}(k(\mathfrak{m}(K))/k) \to G$.

A.2 The relation between $\Phi_{K/k}$ and $\mathfrak{s}_{K/k,S,p,r}$

As in §2.2.1 we write $S^0 = S^0_{K/k}$ for the set of places of k comprising all archimedean places and all which ramify in K/k and write $S = S_{K/k}$ for the union of S^0 and the set of places which lie above p. Then the basic properties of Tate's Theta function imply that for each integer s with $s > 1$ one has

$$\begin{aligned}\Theta_{K/k,S}(s) &= \Theta_{K/k,S^0}(s) \prod_{\substack{v \mid p \\ v \text{ unram}}} \left(1 - \mathrm{N}v^{-s} f_v^{-1}\right) \\ &= \Theta_{K/k,S^0}(s) \prod_{\substack{v \mid p \\ v \text{ unram}}} \mathrm{N}v^{-s} \left(\mathrm{N}v^{s} - f_v^{-1}\right).\end{aligned} \tag{A.2}$$

For each archimedean place v of k define a $\mathbb{C}[G]$-valued function $C_v(s)$ on $\mathbb{C}$ by

$$C_v(s) := \begin{cases} e^{i\pi s} - e^{i\pi s} c_v & \text{if } v \text{ is complex} \\ e^{i\pi s/2} + e^{-i\pi s/2} c_v & \text{if } v \text{ is real} \end{cases}$$

where c_v is the unique generator of the decomposition subgroup of G associated to v. Then by combining the relation (A.2) with [56, Thm. 2.1] we obtain the following relation between the meromorphic $\mathbb{C}[G]$-valued functions $\Theta_{K/k,S}(s)$ and $\Phi_{K/k}(1-s)$ of $s \in \mathbb{C}$

$$\begin{aligned} & i^{r_2(k)} \sqrt{d_k} \left(\prod_{\substack{v \mid p \\ v \text{ unram}}} \mathrm{N}v^{-s} \left(\mathrm{N}v^{s} - \sigma_v^{-1}\right) \right) \Phi_{K/k}(1-s) \\ &\qquad = \left((2\pi)^{-s}\Gamma(s)\right)^d \left(\prod_{v \mid \infty} C_v(s) \right) \Theta_{K/k,S}(s). \end{aligned} \tag{A.3}$$

Now since k is totally real and $r \neq 1$ is integral, for each archimedean place v one has $C_v(r) = i^r(1 + (-1)^r \tau) = 2i^r e_r^-$. Thus, under the hypothesis that K is CM and k

is totally real, (A.3) implies that at $s = r$ we have

$$\begin{aligned}
&\sqrt{d_k}\left(\prod_{\substack{v|p\\ v \text{ unram}}} \mathrm{N}v^{-r}\left(\mathrm{N}v^{r} - \sigma_v^{-1}\right)\right)\Phi_{K/k}(1-r)\\
=&((2\pi)^{-r}(r-1)!)^{d}\Big(\prod_{v|\infty}(2i^{r}e_r^{-})\Big)\Theta_{K/k,S}(r)\\
=&\frac{1}{(2^{r-1}(r-1)!)^{d}}\left(\frac{i}{\pi}\right)^{rd} e_r^{-}\Theta_{K/k,S}(r).
\end{aligned}$$

Since $r \neq 1$ and $\Phi_{K/k}$ is regular away from 0 one therefore has

$$a^{-}_{K/k,S,r} = (2^{r-1}(r-1)!)^{d}\left(\prod_{\substack{v|p\\ v \text{ unram}}} \mathrm{N}v^{-r}\left(\mathrm{N}v^{r} - \sigma_v^{-1}\right)\right)\sqrt{d_k}\Phi_{K/k}(1-r).$$

A.3 The independence of $\mathfrak{s}_{K/k,S,p,r}$ from the choice of j

Every element α of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ induces an automorphism of $\mathbb{Q}\left(\mu_{f(K)}\right)[G]$ by its action on coefficients. To describe its action on $\Phi_{K/k}(1-r)$, we let $\mathbb{Q}^{ab}$ and k^{ab} denote the maximal abelian extensions of $\mathbb{Q}$ and k respectively inside $\overline{\mathbb{Q}}$ and write ver for the transfer homomorphism $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \to \mathrm{Gal}(k^{ab}/k)$. If F is any extension of k within k^{ab} we compose ver with the restriction map to get a homomorphism

$$V_F : \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \to \mathrm{Gal}(F/k)$$

exactly as in [56, §3].

Lemma A.3.1. *For each α in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ one has*

$$\alpha(\Phi_{K/k}(1-r)) = V_K\left(\alpha\mid_{\mathbb{Q}^{ab}}\right)\Phi_{K/k}(1-r)$$

where the product on the right hand side is in $\mathbb{Q}\left(\mu_{f(K)}\right)[G]$.

Proof. For α as above it follows from [53, Prop. 3.1] that

$$\alpha(\Phi_{\mathfrak{m}(K)}(1-r)) = V_{k(\mathfrak{m}(K))}\left(\alpha\mid_{\mathbb{Q}^{ab}}\right)\Phi_{\mathfrak{m}(K)}(1-r)$$

and so by applying $\left(|d_k|N\mathfrak{f}(K)\right)^{-r}\pi_{k(\mathfrak{m}(K)),K}$ to both sides we get the desired result. $\square$

Lemma A.3.2. *For any* $\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *and* $\theta \in \bigwedge^d_{\mathbb{Z}[G]} P^1\left(\mathcal{O}_{K,S}, \mathbb{Z}_p(r)\right)$, *we have*

$$(j\circ\alpha)(\sqrt{d_k})R^{(j\circ\alpha)}_{K/k,p,r}(\theta) = V_K\left(\alpha\mid_{\mathbb{Q}^{ab}}\right)j(\sqrt{d_k})R^{(j)}_{K/k,p,r}(\theta).$$

The element $j(\sqrt{d_k})R^{(j)}_{K/k,p,r}(\theta)$ *belongs to* $\mathbb{Q}_p\left(\mu_{f(K)}\right)[G]$.

Proof. The proof of the first claim here is identical to that of the second claim of [53, Prop 3.2]. To proceed we set $d := [k:\mathbb{Q}]$, write $\tau_1,\dots,\tau_d$ for the embeddings $k\hookrightarrow\overline{\mathbb{Q}}$, and let $\{\widetilde{\tau}_i : 1\le i\le d\}$ be fixed elements of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which extend the given embeddings. We choose $\{\widetilde{\tau}_i : 1\le i\le d\}$ to constitute a complete set of representatives of the coset space $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\mathrm{Gal}(\overline{\mathbb{Q}}/k)$.

Given $\alpha\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ there exists elements $\gamma_1,\dots,\gamma_d$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ and a permutation π_α of $\{1,\dots,d\}$ such that for each i one has

$$\alpha\circ\widetilde{\tau}_i = \widetilde{\tau}_{\pi_\alpha(i)}\circ\gamma_i.$$

In particular, $\mathrm{ver}(\alpha|_{\mathbb{Q}^{\mathrm{ab}}})$ is by definition equal to the image of $\prod_{i=1}^{i=d}\gamma_i$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/k)^{\mathrm{ab}}$. Thus one has

$$V_K\left(\alpha|_{\mathbb{Q}^{\mathrm{ab}}}\right) = \prod_{i=1}^{d}\gamma_i|_K.$$

Recall the isomorphism $\log_{p,r}^{BK}$ from (5.6) and the maps $\lambda_{i,p,r}^{(j)}$ described just prior to Definition 5.3.1. For any $a \in P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ and each $i = 1, \ldots, d$ the definition of $\lambda_{i,p,r}^{(j)}$ implies that

$$\begin{aligned}
\lambda_{i,p,r}^{(j\circ\alpha)}(a \otimes 1) &= \sum_{g\in G} j \circ \alpha \left(\widetilde{\tau}_i \left(g(\log_{p,r}^{\mathrm{BK}}(a))\right)\right) g^{-1} \\
&= \sum_{g\in G} j \left(\widetilde{\tau}_{\pi_\alpha(i)} \left(\gamma_i g(\log_{p,r}^{\mathrm{BK}}(a))\right)\right) g^{-1} \\
&= \sum_{g\in G} j \left(\widetilde{\tau}_{\pi_\alpha(i)} \left(g(\log_{p,r}^{\mathrm{BK}}(a))\right)\right) \left(g \circ \gamma_i^{-1}|_K\right)^{-1} \\
&= \lambda_{\pi_\alpha(i),p,r}^{(j)}(a \otimes 1)\gamma_i|_K
\end{aligned}$$

and hence by linearity one has

$$R_{K/k,p,r}^{(j\circ\alpha)}(\theta) = \mathrm{sgn}(\pi_\alpha) \left(V_K \left(\alpha|_{\mathbb{Q}^{\mathrm{ab}}}\right)\right) R_{K/k,p,r}^{(j)}(\theta)$$

for each element θ of $\bigwedge_{\mathbb{Z}[G]}^d P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$. Also $j \circ \alpha\left(\sqrt{d_k}\right) = \mathrm{sgn}(\pi_\alpha) j(\sqrt{d_k})$ and so

$$(j \circ \alpha)(\sqrt{d_k}) R_{K/k,p,r}^{(j\circ\alpha)}(\theta) = V_K\left(\alpha\,|_{\mathbb{Q}^{ab}}\right) j(\sqrt{d_k}) R_{K/k,p,r}^{(j)}(\theta).$$

This proves the first claim of the Lemma and the second claim then follows directly from the first claim and the class-field theoretic fact that $V_K = \pi_{k(\mathfrak{m}(K)),K} \circ V_{k(\mathfrak{m}(K))}$ factors through the restriction map $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \to \mathrm{Gal}\left(\mathbb{Q}\left(\mu_{f(K)}\right)/\mathbb{Q}\right)$. $\square$

Proposition A.3.3. *For any θ in $\bigwedge_{\mathbb{Z}[G]}^d P^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(r))$ the element*

$$j\left(\sqrt{d_k}\,\Phi_{K/k}(1-r)^{\#}\right) R_{K/k,p,r}^{(j)}(\theta) \in \overline{\mathbb{Q}}_p[G]$$

belongs to $\mathbb{Q}_p[G]$ and is independent of the choice of j.

Proof. The previous two lemmas show that this element is unchanged when j is replaced by $j \circ \alpha$ for any $\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence the element is indeed independent of the choice of j.

For the containment in $\mathbb{Q}_p[G]$ we note that the Bloch-Kato exponential map, and hence also the Bloch-Kato logarithm map $\log_{p,r}^{\mathrm{BK}}$, is p-adically continuous and hence that any element β of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ commutes with it. Thus letting β act on coefficients we find that

$$\begin{aligned}\beta\left(j\left(\sqrt{d_k}\Phi_{K/k}(1-r)^{\#}\right)R^{(j)}_{K/k,p,r}(\theta)\right) &= (\beta\circ j)\left(\sqrt{d_k}\Phi_{K/k}(1-r)^{\#}\right)R^{(\beta\circ j)}_{K/k,p,r}(\theta)\\ &= j\left(\sqrt{d_k}\Phi_{K/k}(1-r)^{\#}\right)R^{(j)}_{K/k,p,r}(\theta).\end{aligned}$$

Hence the element is invariant under all such β and so must be contained in $\mathbb{Q}_p[G]$. $\square$

By combining the last result with the expression for $a^{-}_{K/k,S,r}$ given at the end of §A.2 one finds that the map $\mathfrak{s}^{(j)}_{K/k,S,p,r}$ is independent of the choice of j and that its image $\mathfrak{S}^{(j)}_{K/k,S,p,r}$ is contained in $\mathbb{Q}_p[G]$, as required.

Appendix B

Relating p-adic and complex L-functions

This section is joint work with David Burns.

Let K be a CM abelian extension of a totally real field k. In this section we conjecture, for each integer r strictly greater than one, a precise relationship between the leading terms at $s = r$ of the equivariant complex and p-adic L-functions that are associated to K/k. This conjecture is a natural analogue of Serre's p-adic Stark Conjecture at $s = 1$ and gives a precise (conjectural) connection between Theorem 4.2.1 and the appropriate special case of the ETNC.

B.1 Statement of the conjecture

We fix an integer r that is strictly greater than one and set $\mathfrak{A}^+ := \mathbb{Z}_p[G]e_r^+$ and $A^+ := \mathfrak{A}^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p[G]e_r^+$. In the following conjecture we use the isomorphisms $\vartheta_{p,r}$ and $\vartheta_{\infty,r}$ and graded $\mathbb{Q}[G]$-module $\Xi(\mathbb{Q}(r)_K)$ that are introduced in §5.2 and the S-truncated twisted equivariant p-adic L-function $\mathfrak{L}_{p,S}(s)$ that is introduced in §2.2.3. We note that if Schneider's conjecture (Conjecture 3.2.9) is valid for K at p and r, then Proposition 4.4.2 implies that the complex $R\Gamma(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))e_r^+$ is acyclic and hence induces an identification

$$\left[R\Gamma_c(\mathcal{O}_{K,S}, \mathbb{Q}_p(r))e_r^+\right]_{A^+} \cong (A^+, 0). \tag{B.1}$$

Conjecture B.1.1 (The "p-adic Stark Conjecture at $s = r$" for $r > 1$)**.**

(i) Schneider's conjecture is valid for K at p and r.

(ii) $\vartheta_{\infty,r}^{-1}((L_S^*(\mathbb{Q}(r)_K, 0)^{-1}, 0))$ *belongs to* $\Xi(\mathbb{Q}(r)_K)$*.*

(iii) There exists a unit u of $\mathfrak{A}^+$ such that, with respect to the identification (B.1), one has

$$\vartheta_{p,r} \circ \vartheta_{\infty,r}^{-1}((e_r^+ L_S^* (\mathbb{Q}(r)_K, 0), 0)) = (u\mathfrak{L}_{p,S}(r), 0).$$

Remark B.1.2. By unwinding the definitions of $\vartheta_{p,r}$ and $\vartheta_{\infty,r}$ it is possible to rephrase the equality of Conjecture B.1.1(iii) as asserting that $e_r^+ L_S^* (\mathbb{Q}(r)_K, 0)$ is equal to $u\mathfrak{L}_{p,S}(r)$ multiplied by a product of an explicit archimedean regulator (constructed using the Beilinson regulator) and an explicit p-adic regulator (constructed using the Bloch-Kato logarithm map). In this way one finds that Conjecture B.1.1 is in fact a very natural analogue of the "p-adic Stark Conjecture at $s = 1$" which originates with Serre but is first described precisely by Burns and Venjakob in [16, §5.2] and this analogy suggests that it might also be reasonable to expect that the unit u in Conjecture B.1.1(iii) is very simple, possibly even just e_r^+. In some very recent work Besser, Buckingham, de Jeu and Roblot [5] have also formulated a generalisation to values of L-functions at integers strictly greater than one of the p-adic Stark conjecture at $s = 1$. Their methods are different from those used here and it would surely be interesting to explore the explicit connection between their conjecture and Conjecture B.1.1.

B.2 The relation to the ETNC

Proposition B.2.1. *Let K be a CM abelian extension of a totally real field k and r be an integer strictly greater than one. Assume that K contains a primitive p^{th}-root of* *unity and that* $\mu(K,p)=0$. *Then the validity of Conjecture B.1.1 implies the validity of* $\mathrm{ETNC}^{(j),+}(\mathbb{Q}(r)_K,\mathbb{Z}_p[G])$.

Proof. If Conjecture B.1.1(i) is valid, then Schneider's conjecture is valid for K at r and p and so Proposition 4.4.2(iv) implies that the idempotent $e_r^{(0)}$ defined in Theorem 4.2.1 is equal to e_r^+. Under the stated conditions, Theorem 4.2.1(iii) therefore gives an equality of graded $\mathfrak{A}^+$-modules

$$\left[R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^+\right]_{\mathfrak{A}^+}=\left(\mathfrak{A}^+\mathfrak{L}_{p,S}(r)^{-1},0\right).$$

Since the element u in Conjecture B.1.1(iii) is assumed to be a unit of $\mathfrak{A}^+$ this equality combines with that of Conjecture B.1.1(iii) to imply that

$$\left[R\Gamma_c(\mathcal{O}_{K,S},\mathbb{Z}_p(r))e_r^+\right]_{\mathfrak{A}^+}=\mathfrak{A}^+\cdot\vartheta_{p,r}\circ\vartheta_{\infty,r}^{-1}((e_r^+L_S^*\left(\mathbb{Q}(r)_K,0\right)^{-1},0)).$$

But this equality is just the e_r^+-part of the equality of Proposition 5.2.5 and hence is equivalent to the statement of $\mathrm{ETNC}^{(j),+}(\mathbb{Q}(r)_K,\mathbb{Z}_p[G])$. $\square$

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